Table of Contents

A Short Review of Basic Calculus

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Version
This is version 1, the most up to date version will be at http://www.lloydramey.com/calc_review.pdf.
It was released in Nov. 2014.

Purpose/Introduction
One of my assignments for AP calculus was to make some sort of review for the AP test. The teacher suggested things like flash cards, but said that anything which could help us review would qualify. The assignment was called Calculus At A Glance, it's entirely possible that you know of it.

A friend who had the same assignment the previous year had compiled all of her notes (which were and have always been very thorough) into a binder. I knew that the compilation effort would be more helpful to me than any studying of flash cards or notes, and so I liked the idea. I did not however like the idea of handwriting something that could easily fill a .5" binder. This is the result of a weekends worth of work.

If you find any typos, or even worse actual mathematical mistakes, please shoot me an email.

Last spring, I lent this to a friend taking intro level calculus. He said that I should put this on the web because people might actually find this useful. I've finally gotten around to it, and I hope that it helps someone.

1.1 Analytic and Geometric Methods of Determining Odd or Even Functions

Analytic:

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.

For example: $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

If a function f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function.

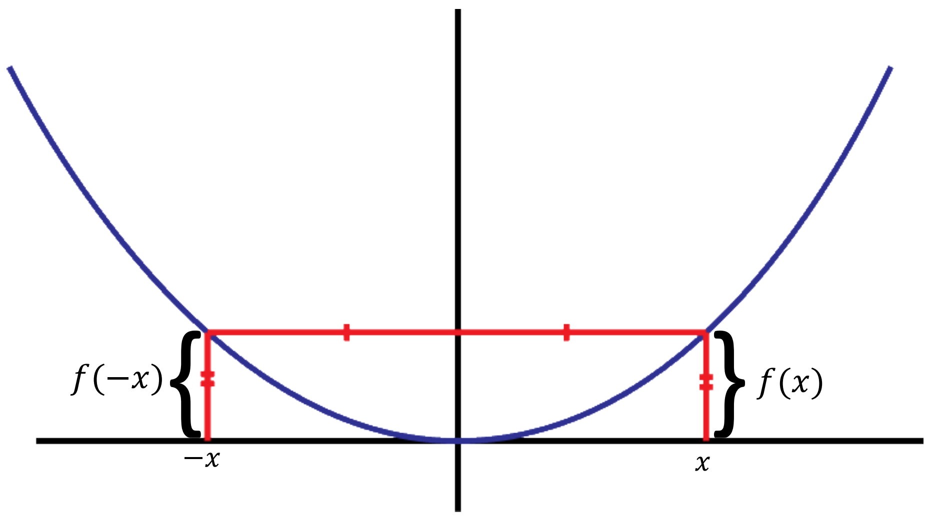
For example: $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

Geometric:

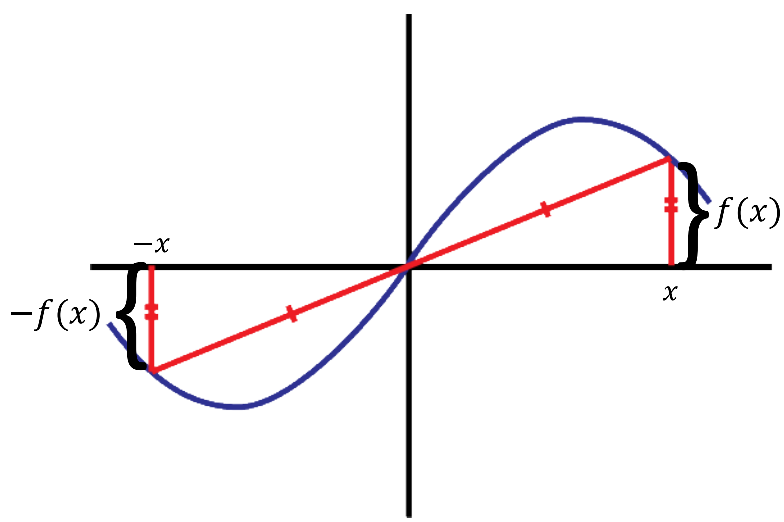
The geometric significance of an even function is that its graph is symmetric with respect to the y-axis.

For example:



The graph of an odd function is symmetric about the origin

For example:



1.2 Definition of e (e expressed as a limit)

e is defined as the number such that $\ln e = 1$

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e \quad \text{OR} \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Proof: Let $f(x) = \ln x$. Then $f'(x) = \frac{1}{x}$, so $f'(1) = 1$. But by the definition of the derivative,

$$f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} = \lim_{x\to 0}\frac{f(1+x) - f(1)}{x}$$

$$= \lim_{x\to 0}\frac{\ln(1+x) - \ln 1}{x} = \lim_{x\to 0}\frac{1}{x}\ln(1+x)$$

$$= \lim_{x\to 0}\ln(1+x)^{\frac{1}{x}} = \ln\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right]$$

Because $f'(1) = 1$, we have

$$\ln\left[\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right] = 1$$

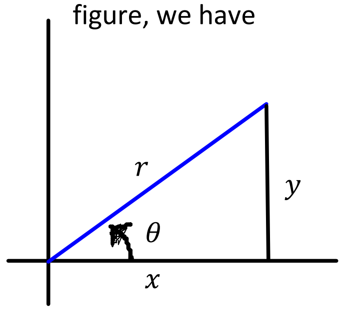
Therefore

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

1.3 Coordinate Conversion Between Polar and Rectangular(Cartesian)

Polar to rectangular:

If the point P has the Cartesian coordinates (x, y) and the polar coordinates (r, θ), then, from the figure, we have



$\cos\theta = \frac{x}{r}$ $\quad\quad$ $\sin\theta = \frac{y}{r}$

And so

$x = r\cos\theta$ $\quad\quad$ $y = r\,sin\,\theta$

Example:

Convert the point $\left(2, \frac{\pi}{3}\right)$ from polar to Cartesian coordinates.

$$r = 2 \qquad\qquad \theta = \frac{\pi}{3}$$

$$x = r\cos\theta = 2\cos\left(\frac{\pi}{3}\right) = 2 \cdot \frac{1}{2} = 1$$

$$y = r\,sin\,\theta = 2\sin\left(\frac{\pi}{3}\right) = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$$

Therefore, the point is $(1, \sqrt{3})$ in Cartesian Coordinates.

Rectangular to polar:

To find r and θ when x and y are known, we use these equations:

$$r^2 = x^2 + y^2 \qquad\qquad \tan\theta = \frac{y}{x}$$

Example:

Represent the point with Cartesian coordinates $(1, -1)$ in terms of polar coordinates.

If we choose r to be positive, then

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\tan\theta = \frac{y}{x} = -1$$

Since the point $(1, -1)$ lies in the fourth quadrant, we can choose $\theta = -\frac{\pi}{4}$ or $\theta = \frac{7\pi}{4}$. Thus, one possible answer is $\left(\sqrt{2}, -\frac{\pi}{4}\right)$; another is $\left(\sqrt{2}, \frac{7\pi}{4}\right)$.

2.1 - Basic Properties of Limits

Limit Laws:

Suppose that c is a constant, and the limits

$\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$

exist. Then

1. $\lim\limits_{x\to a}[f(x)+g(x)] = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$
2. $\lim\limits_{x\to a}[f(x)-g(x)] = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$
3. $\lim\limits_{x\to a}[cf(x)] = c\lim\limits_{x\to a} f(x)$
4. $\lim\limits_{x\to a}[f(x)g(x)] = \lim\limits_{x\to a} f(x)\cdot\lim\limits_{x\to a} g(x)$
5. $\lim\limits_{x\to a}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$

Example:

(a) $\lim_{x\to -2}[f(x)+5g(x)]$ (b) $\lim_{x\to 1}[f(x)g(x)]$ (c) $\lim_{x\to 2}\left[\frac{f(x)}{g(x)}\right]$

$\lim_{x\to -2} f(x) = 1$ $\lim_{x\to -2} g(x) = -1$

$\lim_{x\to 1} f(x) = 2$ $\lim_{x\to 1} g(x) = DNE$

$\lim_{x\to 2} f(x) \approx 1.4$ $\lim_{x\to 2} g(x) = 0$

Solutions:

(a) $\lim_{x\to -2}[f(x)+5g(x)] = \lim_{x\to -2} f(x) + \lim_{x\to -2} 5g(x)$ (Law 1)

$= \lim_{x\to -2} f(x) + 5\lim_{x\to -2} g(x)$ (Law 2)

$= 1 + 5(-1) = 4$

(b) Because $\lim_{x\to 1} g(x) = DNE$, the given limit does not exist.

(c) Because $lim_{x\to 1} g(x) = 0$, the given limit does not exist.

6. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$ (where n is a positive integer)

Special Limit Laws (using the following two special limits)

7. $\lim\limits_{x\to a} c = c$
8. $\lim\limits_{x\to a} x = a$
9. $\lim_{x\to a} x^n = a^n$ (where n is a positive integer)
10. $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$ (If n is even, we assume that $a > 0$)
11. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$ (If n is even, we assume that $\lim_{x\to a} f(x) > 0$)

Example:

$$\lim_{x \to -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} = \frac{\lim_{x \to -2} (x^3 + 2x^2 - 1)}{\lim_{x \to -2} (5 - 3x)}$$

$$= \frac{\lim_{x \to -2} x^3 + 2 \lim_{x \to -2} x^2 + \lim_{x \to -2} 1}{\lim_{x \to -2} 5 + 3 \lim_{x \to -2} x}$$

$$= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)}$$

$$= \frac{1}{11}$$

2.2 - Simplifying $\frac{0}{0}$

Factoring:

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}x+1=2$$

$$F(h)=\frac{(3=h)^2-9}{h}=\frac{(9+6h+h^2)-9}{h}=\frac{6h+h^2}{h}=6+h$$

$$\underset{h\to 0}{lim}\frac{(3+h)^2-9}{h}=\underset{h\to 0}{lim}(6+h)=6$$

Multiplying by conjugate:

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}=\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}=\lim_{x\to 0}\frac{(t^2+9)-9}{t^2\left(\sqrt{t^2+9}+3\right)}=\lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$=\underset{t\to 0}{lim}\frac{1}{\sqrt{t^2+9}+3}=\frac{1}{\sqrt{\underset{t\to 0}{lim}(t^2+9)}+3}=\frac{1}{3+3}=\frac{1}{6}$$

Common Denominators:

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{x-2}=\lim_{x\to 2}\frac{\frac{2-x}{2x}}{x-2}=\lim_{x\to 2}-\frac{\frac{x-2}{2x}}{x-2}=\lim_{x\to 2}-\frac{x-2}{2x}\cdot\frac{1}{x-2}=\lim_{x\to 2}-\frac{1}{2x}=-\frac{1}{4}$$

2.3 - How to Find Horizontal and Slant Asymptotes Analytically

Limits at positive infinity:

Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Limits at negative infinity:

Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L$$

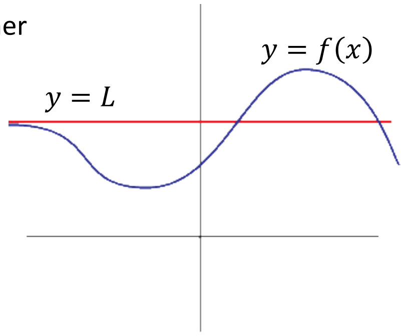
means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large negative.

Note: The symbol ∞ does not represent a number. Nonetheless, $x \to \pm\infty$ is often read "as x approaches positive or negative infinity."

Horizontal asymptotes:

The line $y = L$ is called a horizontal asymptote of the curve $y = f(x)$ if either

$\lim_{x\to\infty} f(x) = L$ OR $\lim_{x\to-\infty} f(x) = L$



If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

Example:

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} = \frac{\lim_{x\to\infty}\left(3 - \frac{1}{x} - \frac{2}{x^2}\right)}{\lim_{x\to\infty}\left(5 + \frac{4}{x} + \frac{1}{x^2}\right)}$$

$$= \frac{\lim_{x\to\infty} 3 - \lim_{x\to\infty} \frac{1}{x} - 2\lim_{x\to\infty} \frac{1}{x^2}}{\lim_{x\to\infty} 5 + 4\lim_{x\to\infty} \frac{1}{x} + \lim_{x\to\infty} \frac{1}{x^2}}$$

$$= \frac{3 - 0 - 0}{5 + 0 + 0} = \frac{3}{5}$$

$\lim_{x\to-\infty} f(x) = \frac{3}{5}$ as well.

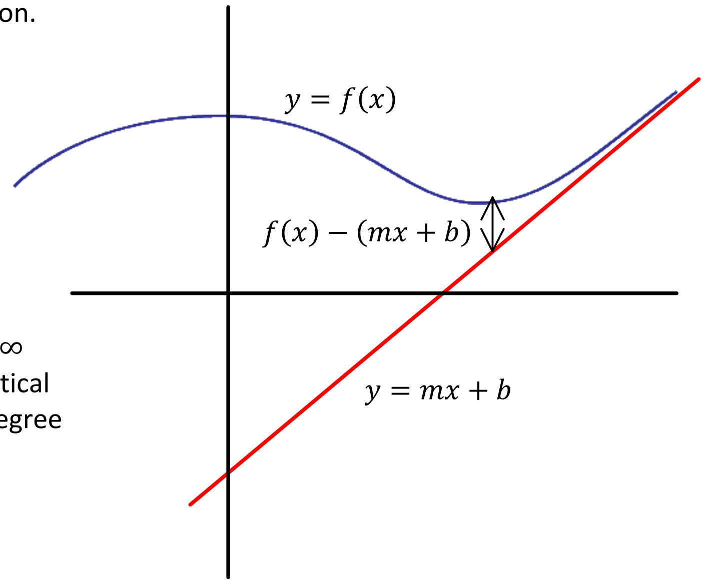
This shows that there is a horizontal asymptote at $y = \frac{3}{5}$ or the function $f(x) = \frac{3x^2-x-2}{5x^2+4x+1}$.

Slant asymptotes:

Some curves have asymptotes that are *oblique*, that is, neither horizontal nor vertical. If

$$\lim_{x\to\infty}[f(x)-(mx+b)]=0$$

then the line $y=mx+b$ is called a slant asymptote because the vertical distance between the curve $y=f(x)$ and $y=mx+b$ approaches 0, as in the figure below. For rational functions, slant asymptotes occur when the degree of the numerator is one more than that of the denominator. In such a case, the equation of the slant asymptote can be found by long division.



Example:

$$f(x)=\frac{x^3}{x^2+1}$$

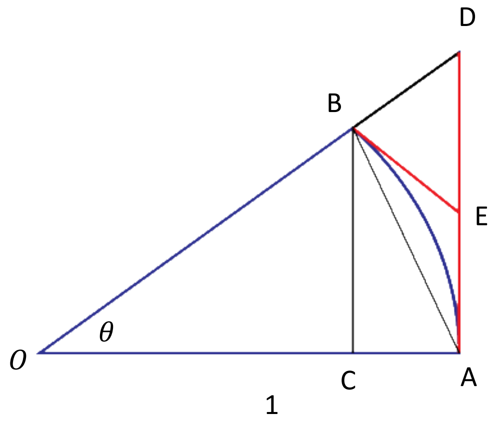
Since x^2+1 is never negative, there is no vertical asymptote. Since $f(x)\to\infty$ as $x\to\infty$ and $f(x)\to-\infty$ as $x\to-\infty$, there is no vertical asymptote. But long division (because the degree of the numerator is larger than that of the denominator) gives

$$f(x)=\frac{x^3}{x^2+1}=x-\frac{x}{x^2+1}$$

$$f(x)-x=-\frac{x}{x^2+1}=\frac{\frac{1}{x}}{1+\frac{1}{x^2}}\to 0 \quad \text{as} \quad x\to\pm\infty$$

So, the line $y=x$ is a slant asymptote.

2.4 - Two Special Trigonometric Limits

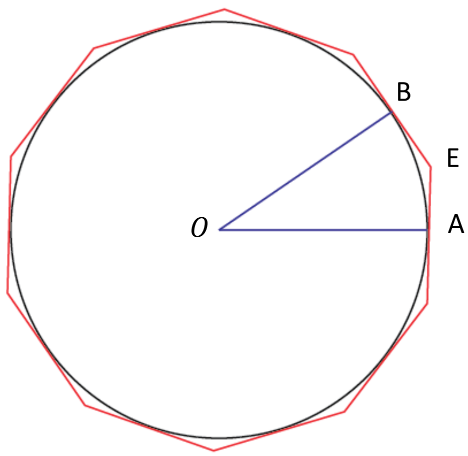


$\lim_{x\to 0}\frac{\sin\theta}{\theta} = 1$ because:

Assume that θ lies between 0 and $\frac{\pi}{2}$. The figure to the right shows a sector of the circle with center O, angle θ, and a radius of 1.

$arc\ AB = \theta$. Also, $|BC| = |OB|\sin\theta = \sin\theta$

So, $|BC| < |AB| <$ arc AB, making $\sin\theta < \theta$, so $\frac{\sin\theta}{\theta} < 1$



Let the tangents at A and B intersect at E. From the figure to the left, it can be seen that the circumference of the circle is smaller than the length of a circumscribed polygon, and so $arc\ AB < |AE| + |EB|$. Thus

$$\begin{aligned}\theta = arc\ AB &< |AE| + EB| \\ &< |AE| + ED| \\ &= |AD| = |OA|\tan\theta \\ &= \tan\theta\end{aligned}$$

So, $\theta < \frac{\sin\theta}{\cos\theta}$, making $\cos\theta < \frac{\sin\theta}{\theta} < 1$

We know that $\lim_{x\to 0} 1 = 1$ and $\lim_{\theta\to 0}\cos\theta = 1$, so by the Squeeze Theorem, we have

$$\lim_{\theta\to 0^+}\frac{\sin\theta}{\theta} = 1$$

$\frac{\sin\theta}{\theta}$ is an even function, so its right limit must be equal to its left limit, therefore

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

$$\lim_{x\to 0}\frac{\cos\theta - 1}{\theta} = \lim_{\theta\to 0}\left[\frac{\cos\theta - 1}{\theta}\cdot\frac{\cos\theta + 1}{\cos\theta + 1}\right] = \lim_{\theta\to 0}\frac{\cos^2\theta - 1}{\theta(\cos\theta + 1)} = \lim_{\theta\to 0} -\frac{\sin^2\theta}{\theta(\cos\theta + 1)}$$

$$= -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\frac{\sin\theta}{\cos\theta + 1} = -\lim_{\theta\to 0}\frac{\sin\theta}{\theta}\cdot\lim_{\theta\to 0}\frac{\sin\theta}{\cos\theta + 1} = -1\cdot\left(\frac{0}{1+1}\right) = 0$$

Therefore, $\lim_{x\to 0}\frac{\cos\theta - 1}{\theta} = 0$

2.5 - Definition of Continuity

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a)$$

Notice, this definition implicitly requires three things:

1. $f(a)$ is defined (that is, a is in the domain of f)
2. $\lim_{x\to a} f(x)$ exists (so f must be defined on an open interval that contains a)
3. $\lim\limits_{x\to a} f(x) = f(a)$

If f is not continuous at a, we say f is discontinuous at a, or f has a discontinuity at a.

Examples:

(a) $f(x) = \frac{x^2-x-2}{x-2}$ $\quad f(2)$ is undefined, so f is discontinuous at 2

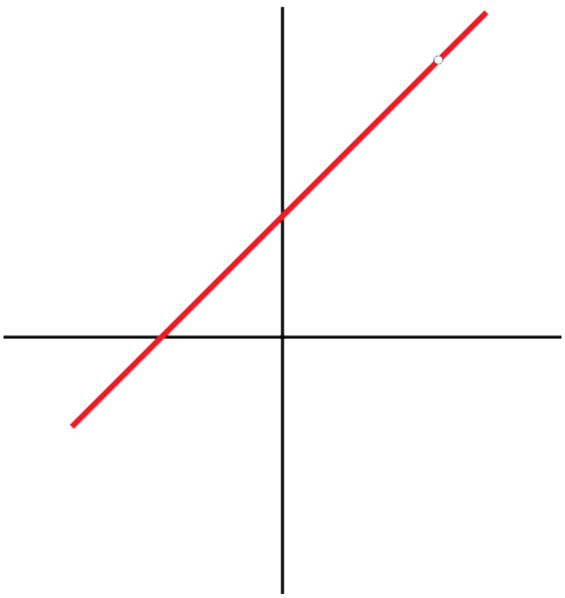
(b) $f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$ $\quad f(0) = 1$ is defined, but $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} = DNE$, so f is discontinuous at 0

(c) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$ $\quad f(2) = 1$ is defined and $\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^2-x-2}{x-2}$

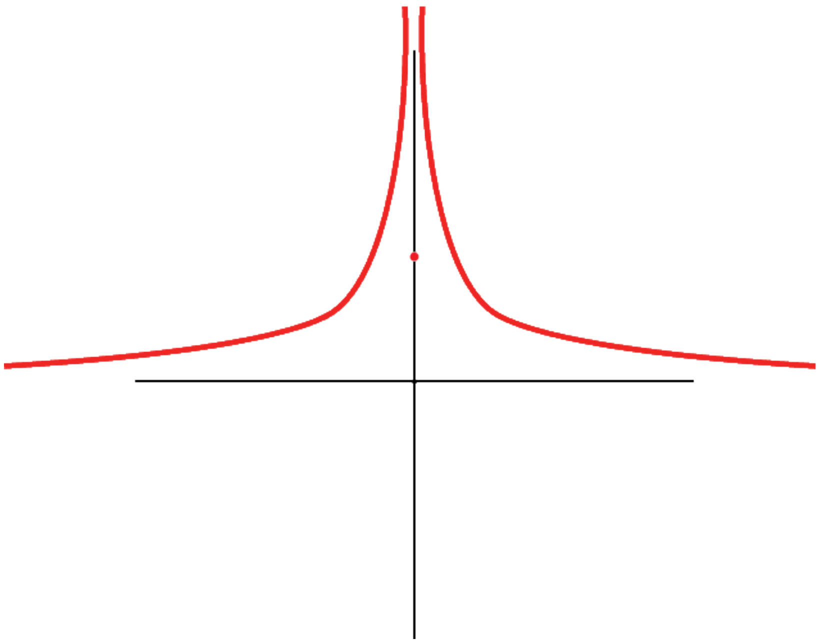
$= lim_{x\to 2} \frac{(x-2)(x+1)}{x-2} = \lim_{x\to 2}(x+1) = 3$ exists, but $\lim_{x\to 2} f(x) \neq f(2)$, so f is discontinuous at 2.

(d) $f(x) = [\![x]\!]$ $\quad$ The greatest integer function has discontinuities at all integers because $\lim_{x\to n} [\![x]\!]$ does not exist if n is an integer.

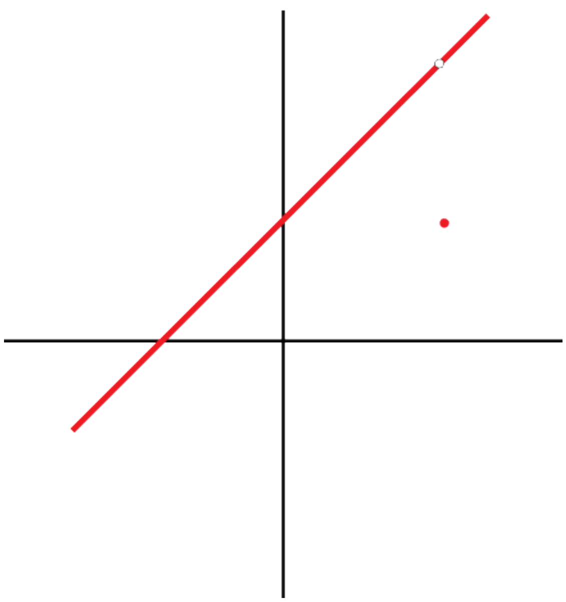
2.6 - Diagrams/Descriptions of Removable and Non-Removable Discontinuities



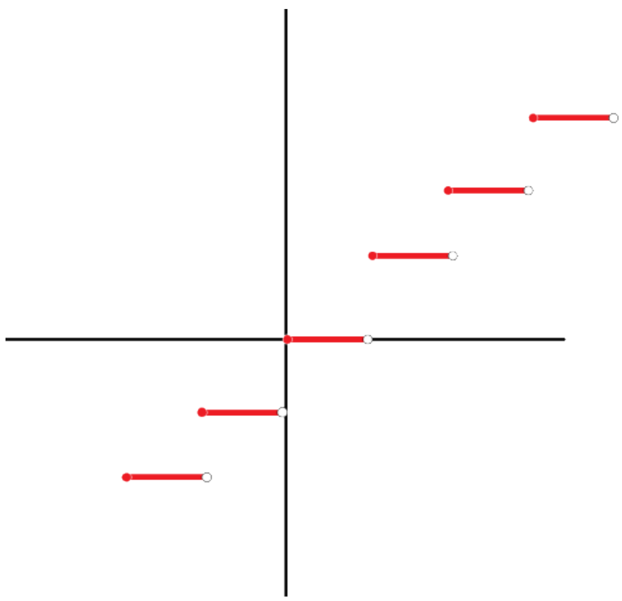
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$



(b) $f(x) = \begin{cases} \frac{1}{x^2}, & x \neq 0 \\ 1, & x = 0 \end{cases}$



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$



(d) $f(x) = [\![x]\!]$

The illustrated discontinuity in both *(a)* and *(c)* is called **removable** because we can remove the discontinuity by redefining f at just the single number 2. [The function $g(x) = x + 1$ is continuous.] The discontinuity in part *(b)* is called an **infinite discontinuity**. The discontinuities in part *(d)* are called **jump discontinuities** because the function "jumps" from one value to another.

Example:

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1,1]$.

If $-1 < a < 1$, then using limit laws, we have

$$\lim_{x\to a} f(x) = \lim_{x\to a}\left(1 - \sqrt{1 - x^2}\right)$$

$$= 1 - \lim_{x\to a}\sqrt{1 - x^2}$$

$$= 1 - \sqrt{\lim_{x\to a}(1 - x^2)}$$

$$= 1 - \sqrt{1 - a^2}$$

$$= f(a)$$

Thus, by the definition of continuity, f is continuous at a if $-1 < a < 1$. Similar calculations show that

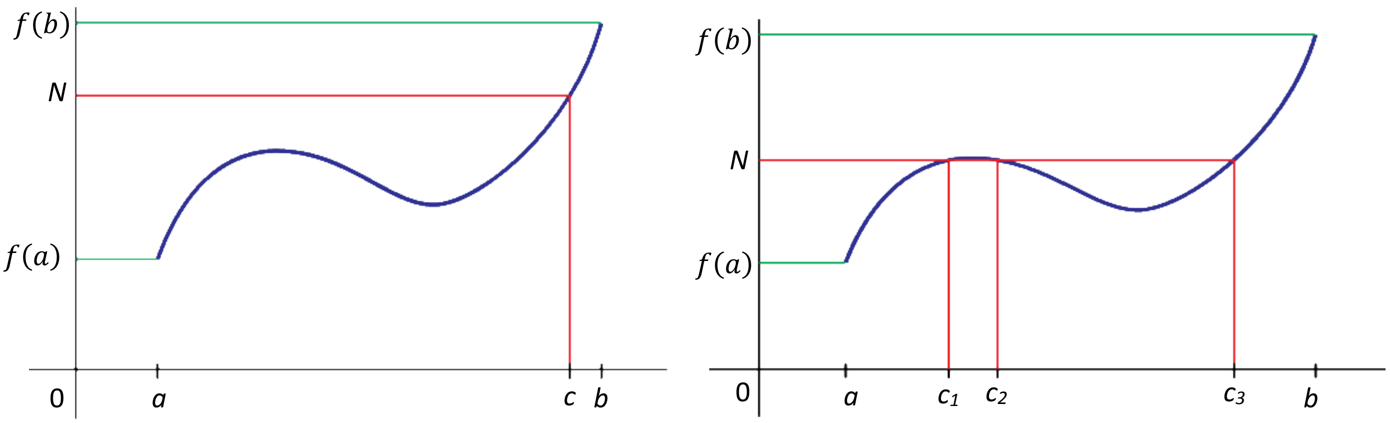
$\lim_{x\to -1^+} f(x) = 1 = f(-1)$ and $\lim_{x\to 1^-} f(x) = 1 = f(1)$

So, f is continuous from the right at -1 and from the left at 1. Therefore, f is continuous on $[-1,1]$

2.7 - Intermediate Value Theorem

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The Intermediate Value Theorem states that a continuous function takes on every intermediate value between the function values $f(a)$ and $f(b)$. It is illustrate by the figures below. Notice that the value N can be taken on once, or more than once.



Example:

Show that there is a root of the equation $4x^3 - 6x^2 + 3x - 2 = 0$ between 1 and 2.

Let $f(x) = 4x^3 - 6x^2 + 3x - 2$. We are looking for a solution of the given equation, that is, a number c between 1 and 2 such that $f(c) = 0$. Therefore, we take $a = 1$, $b = 2$, and $N = 10$. We have

$$f(1) = 4 - 6 + 3 - 2 = -1 < 0$$

and

$$f(2) = 32 - 24 + 6 - 2 = 12 > 0$$

Thus $f(1), 0 < f(2)$, that is $N = 0$ is a number between $f(1)$ and $f(2)$. Now f is continuous since it is a polynomial, so the Intermediate Value Theorem says that there is a number c between 1 and 2 such that $f(c) = 0$. In other words, the equation $4x^3 - 6x^2 + 3x - 2 = 0$ has at least one root c in the interval $(1,2)$.

2.8 - L'Hospital's Rule

Indeterminate quotients ($\frac{0}{0}$ and $\frac{\infty}{\infty}$)

Suppose that f and g are differentiable and $g'(x) \neq 0$ near *a* (except possibly at *a*). Suppose that

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$

or that

$\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$

(In other words, we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$). Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is $\pm\infty$)

Note: L'Hospital's Rule states that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied. It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.

Note: L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the following symbols: $x \to a^+$, $x \to a^-$, $x \to \infty$, $x \to -\infty$.

Example:

Find $\lim_{\mathrm{x}\to 1} \frac{\ln \mathrm{x}}{\mathrm{x}-1}$.

Since

$\lim_{x\to 1} \ln x = \ln 1 = 0$ and $\lim_{x\to 1}(x-1) = 0$

we can apply l'Hospital's Rule:

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = 1$$

Calculate $\lim_{x\to\infty} \frac{e^x}{x^2}$.

We have $\lim_{x\to\infty} e^x = \infty$ and $\lim_{x\to\infty} x^2 = \infty$, so l'Hospital's Rule givess

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x}$$

Since $e^x \to \infty$ and $2x \to \infty$, the limit on the right side is also indeterminate, but a second application of l'Hospital's rule gives

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

Indeterminate products $(0 \cdot \infty)$

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then it isn't clear what the value f $\lim_{x\to a} f(x)g(x)$, if any, will be. There is a struggle between f and g. If f wins, the answer will be 0; if g wins, the answer will be $\pm\infty$. Or there may be a compromise where the answer is a finite nonzero number. This kind of limit is called an **indeterminate form of type $\mathbf{0} \cdot \infty$**. We can deal with it by writing the product fg as a quotient:

$$fg = \frac{f}{\frac{1}{g}} \qquad \text{or} \qquad fg = \frac{g}{\frac{1}{f}}$$

This converts the given limit into an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use l'Hospital's Rule.

Example:

Evaluate $\lim_{x\to 0^+} x \ln x$.

The given limit is indeterminate because, as $x \to 0^+$, the first factor (x) approaches 0 while the second factor $(\ln x)$ approaches $-\infty$. Writing $x = \frac{1}{\left(\frac{1}{x}\right)}$, we have $\frac{1}{x} \to \infty$ as $x \to 0^+$, so l'Hospital's Rule gives

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} (-x) = 0$$

) Note: In solving this example, another possible option would have been to write

$$\lim_{x\to 0^+} x \ln x = \lim_{x\to 0^+} \frac{x}{\frac{1}{\ln x}}$$

This gives an indeterminate form of type $\frac{0}{0}$, but if we apply l'Hospital's Rule we get a more complicated expression than the one we started with. In general, when we rewrite an indeterminate product, we try to choose the option that leads to the simpler limit.

Indeterminate differences $(\infty - \infty)$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a} [f(x) - g(x)]$$

is called an **indeterminate form of type** $\infty - \infty$. Again there is a contest between f and g. Will the answer be ∞ (f wins) or will it be $-\infty$ (g wins) or will the compromise on a finite number? To find out, we try to convert the difference into a quotient (for instance, by using a common denominator or rationalization, or factoring out a common factor) so that we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

Example:

Compute $\lim_{x\to\left(\frac{\pi}{2}\right)^-}(\sec x - \tan x)$.

First notice that $\sec x \to \infty$ and $\tan x \to \infty$ as $x \to \left(\frac{\pi}{2}\right)^-$, so the limit is indeterminate. Here we use a common denominator:

$$\lim_{x\to\left(\frac{\pi}{2}\right)^-}(\sec x - \tan x) = \lim_{x\to\left(\frac{\pi}{2}\right)^-}\left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{1-\sin x}{\cos x}$$

$$= \lim_{x\to\left(\frac{\pi}{2}\right)^-}\frac{-\cos x}{-\sin x} = 0$$

Note that the use of l'Hospital's Rule is justified because $1 - \sin x \to 0$ and $\cos x \to 0$ as $x \to \left(\frac{\pi}{2}\right)^-$

Indeterminate powers (0^0, ∞^0, and 1^∞)

Several indeterminate forms arise from the limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

1. $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ **type 0^0**
2. $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = 0$ **type ∞^0**
3. $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$ **type 1^∞**

Each of these three cases can be treated either by taking the natural logarithm:

let $\quad y = [f(x)]^{g(x)}$, then $\quad \ln y = g(x)\ln f(x)$

or by writing the function as an exponential:

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

(Recall that both of these methods were used in differentiating such functions.) In either method, we are led to the indeterminate product $g(x)\ln f(x)$, which is of type $0 \cdot \infty$.

Example:

Calculate $\lim_{x\to 0^+}(1+\sin 4x)^{\cot x}$.

First notice that as $x \to 0^+$, we have $1 + 4\sin x \to 1$, so the given limit is indeterminate. Let

$$y = (1+\sin x)^{\cot x}$$

Then

$$\ln y = \ln[(1+\sin 4x)^{\cot x}] = \cot x \ln(1+\sin 4x)$$

So l'Hospital's Rule gives us

$$\lim_{x\to 0^+}\ln y = \lim_{x\to 0^+}\frac{\ln(1+\sin 4x)}{\tan x} = \lim_{x\to 0^+}\frac{\frac{4\cos 4x}{1+\sin 4x}}{\sec^2 x} = 4$$

So far we have computed the limit of $\ln y$, but what we want is the limit of y. To find this, we use the fact that $y = e^{\ln y}$:

$$\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} y = \lim_{x \to 0^+} e^{\ln y} = e^4$$

Example:

Find $\lim_{x \to 0^+} x^x$.

$$x^x = \left(e^{\ln x}\right)^x = e^{x \ln x}$$

Use l'Hospital's:

$$\lim_{x \to 0^+} x \ln x = 0$$

Therefore

$$\lim_{x \to 0^+} x^{\mathrm{x}} = \lim_{x \to 0^+} e^{x \ln x} = e^0 = 1$$

3.1 - Definition of a Derivative and Alternate Limit Form of the Derivative

The **derivative of a function $\boldsymbol{f}$ at a number $\boldsymbol{a}$**, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

If we write $x = a + h$, then $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore, an equivalent way of stating this definition is

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example:

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{[(a+h)^2 - 8(a+h) + 9] - [a^2 - 8a + 9]}{h}$$

$$= \lim_{h \to 0} \frac{a^2 + 2ah + h^2 - 8a - 8h + 9 - a^2 + 8a - 9}{h} = \lim_{h \to 0} \frac{2ah + h^2 - 8h}{h}$$

$$= \lim_{h \to 0} (2a + h - 8) = 2a - 8$$

3.2 - Basic Derivative Rules

Constant function ($f(x) = c$)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c-c}{h} = \lim_{h\to 0} 0 = 0$$

Therefore

$$\frac{d}{dx}(c) = 0$$

Power functions ($f(x) = x^n$)

If $n = 1$, $f(x) = x$ is the line $y = x$ which has a slope of 1, therefore

$$\frac{d}{dx}(x) = 1$$

For $n = 4$ we find the derivative of $f(x) = x^4$ as follows:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^4 - x^4}{h}$$

$$= \mathop{lim}_{h\to 0} \frac{x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4 - x^4}{h} = \lim_{h\to 0} \frac{4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h}$$

$$= \mathop{lim}_{h\to 0}(4x^3 + 6x^2h + 4xh^2 + h^3) = 4x^3$$

Thus, if n is a positive integer,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Proof of the power rule

The formula $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$ can easily be verified.
If $f(x) = x^n$, then

$$f'(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a}$$

$$= \mathop{lim}_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

$$= a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1} = na^{n-1}$$

Therefore, if n is any real number,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Constant multiple rule

If c is a constant and f is differentiable, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x)$$

Proof

Let $g(x) = cf(x)$. Then

$$g'(x) = \lim_{h\to 0}\frac{g(x-h)-g(x)}{h} = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h}$$

$$= \mathop{lim}_{h\to 0} c\left[\frac{f(x+h)-f(x)}{h}\right] = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = cf'(x)$$

Sum rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

Proof

Let $F(x) = f(x) + g(x)$. Then

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0}\frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$$

$$= \mathop{lim}_{h\to 0}\frac{f(x+h)-f(x)}{h} + \mathop{lim}_{h\to 0}\frac{g(x+h)-g(x)}{h} = f'(x) + g'(x)$$

The sum rule can be extended to the sum of any number of functions.

Difference rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

Exponential functions ($f(x) = a^x$)

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} = \lim_{h\to 0}\frac{a^x a^h - a^x}{h} = \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h}$$

as the factor a^x doesn't depend on h,

$$f'(x) = a^x\lim_{h\to 0}\frac{a^h-1}{h}$$

Notice that the limit is the value of the derivative of f at 0, that is,

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0) = \ln a$$

Therefore, if the exponential function $f(x) = a^x$ is differentiable at 0, then it is differentiable everywhere and

$$f'(x) = a^x f'(0)$$

or

$$f'(x) = a^x \ln a$$

From this, we can arrive at the following definition of e

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

Natural exponential function

$$\frac{d}{dx}(e^x) = e^x$$

Examples:

Find $f'(x)$

(a) $f(x) = x$ $\quad f'(x) = 1$

(b) $f(x) = x^6$ $\quad f'(x) = 6x^5$

(c) $f(x) = \frac{1}{x}$ $\quad f'(x) = \frac{d}{dx}(x^{-1}) = \frac{-1}{x^2}$

(d) $f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2\sqrt{x}}$

(e) $f(x) = 3x^4$ $\quad f'(x) = 12x^3$

(f) $f(x) = (x^8 + 12x^5 - 4x^4 + 10x^3 - 6x + 5)$ $\quad f'(x) = 8x^7 + 60x^4 - 16x^3 + 10x^3 - 6$

(g) $f(x) = e^x - x$ $\quad f'(x) = e^x - 1$

(h) $(x) = 5x - 1$ $\quad f'(x) = 5$

Product rule

If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Example:

$f(x) = xe^x$, find $f'(x)$

$$f'(x) = \frac{d}{dx}(xe^x) = x\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x) = xe^x + e^x \cdot 1 = (x+1)e^x$$

Quotient rule

Suppose that f and g are both differentiable functions. If we make the assumption that $F(x) = \frac{f}{g}$ is differentiable, then it is not difficult to find a formula for F' in terms of f' and g'.

Since $F(x) = \frac{f(x)}{g(x)}$, we can write $f(x) = F(x)g(x)$ and apply the product rule:

$$f'(x) = F(x)g'(x) + g(x)F'(x)$$

Solving this equation for $F'(x)$, we get

$$F'(x) = \frac{f(x) - F(x)g(x)}{g(x)} = \frac{f(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)} = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Therefore

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

Example:

Let $y = \frac{x^2+x-2}{x^3+6}$.

Then

$$y' = \frac{(x^3+6)\frac{d}{dx}(x^2+x-2) - (x^2+x-2)\frac{d}{dx}(x^3+6)}{(x^3+6)^2}$$

$$= \frac{(x^3+6)(2x+1) - (x^2+x-2)(3x^2)}{(x^3+6)^2}$$

$$= \frac{(2x^4+x^3+12x+6) - (3x^4+3x^3-6x^2)}{(x^3+6)^2}$$

$$= \frac{-x^4-2x^3+6x^2+12x+6}{(x^3+6)^2}$$

Trigonometric functions

Using the known limits $\lim_{x\to 0}\frac{\sin x}{x} = 1$ and $\lim_{x\to 0}\frac{\cos x - 1}{x} = 0$, we can get the derivative of $f(x) = \sin x$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0}\frac{sin\,x\,cos\,h + cos\,x\,sin\,h - \sin x}{h} = \lim_{h\to 0}\left[\frac{\sin x\cos x - \sin x}{h} + \frac{\cos x\sin x}{h}\right]$$

$$= \lim_{h\to 0}\left[sin\,x\left(\frac{cos\,h - 1}{h}\right) + cos\,x\left(\frac{\sin h}{h}\right)\right]$$

$$= \lim_{h\to 0} sin\,x \cdot \lim_{h\to 0}\frac{(cos\,h - 1)}{h} + \lim_{h\to 0} cos\,x \cdot \lim_{h\to 0}\frac{sin\,h}{h}$$

$$= (sin\,x)\cdot 0 + (cos\,x)\cdot 1 = cos\,x$$

Therefore

$$\frac{d}{dx}(\sin x) = \cos x$$

Example:

Differentiate $y = x^2 \sin x$.

$$\frac{dy}{dx} = x^2\frac{d}{dx}(\sin x) + \sin x\frac{d}{dx}(x^2) = x^2\cos x + 2x\sin x$$

Using similar methods, we arrive at

$$\frac{d}{dx}(\cos x) = -\sin x$$

While the tangent function can also be differentiated by using the deffinition of a derivative, it is easier to use the quotient rule:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\frac{d}{dx}(\sin x) - \sin x\frac{d}{dx}(\cos x)}{\cos^2 x}$$

$$= \frac{cos\,x \cdot cos\,x - sin\,x\,(-\,sin\,x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Therefore

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Other trigonometric derivatives

$$\frac{d}{dx}(\sec x) = \sec x\tan x$$

$$\frac{d}{dx}(\csc x) = -\csc x\cot x$$

$$\frac{d}{dx}(cot\,x) = -\,csc^2\,x$$

Example:

Differentiate $f(x) = \frac{\sec x}{1+\tan x}$.

$$f'(x) = \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x\frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+tan\,x)\,sec\,x\,tan\,x - sec\,x \cdot sec^2\,x}{(1+\tan x)^2}$$

$$= \frac{\sec x\,[\tan x + \tan^2 x - \sec^2 x]}{(1+\tan x)^2}$$

$$= \frac{sec\,x\,(tan\,x - 1)}{1 + tan^2\,x}$$

Inverse trigonometric functions (using implicit differentiations)

$y = \sin^{-1} x$ means $\sin x = y$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

Differentiating $y = \sin^{-1} x$ with respect to x, we obtain

$\cos y \frac{dy}{dx} = 1$ or $\frac{dy}{dx} = \frac{1}{\cos x}$

Now $\cos y \ge 0$, since $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, so

$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$$

Therefore

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

The arctangent function is differentiate in a similar way. If $y = \tan^{-1} x$, then $\tan y = x$. Differentiating this implicitly with respect to x, we have

$$\sec^2 y \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Other inverse trigonometric derivatives

$$\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$$

Example:

Differentiate $y = \frac{1}{\sin^{-1}x}$.

$$\frac{dy}{dx} = \frac{d}{dx}(\sin^{-1}x)^{-1} = -1(\sin^{-1}x)^{-2}\frac{d}{dx}(\sin^{-1}x) = \frac{-1}{(\sin^{-1}x)^2\sqrt{1-x^2}}$$

Other functions

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

3.3 - Chain Rule

If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by

$$F'(x) = f'(g(x))g'(x)$$

In Leibniz notation, if $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Example:

Differentiate $F(x) = \sqrt{x^2+1}$.

$$f(u) = \sqrt{u}$$
$$g(x) = x^2 + 1$$
$$f'(u) = \frac{1}{2}u^{-\frac{1}{2}} \quad \text{and} \quad g'(x) = 2x$$

$$F'(x) = f'(g(x))g'(x) = \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$$

The power rule combined with the chain rule

If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

Alternatively,

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1} \cdot g'(x)$$

Example:

Find $f'(x)$ if $f(x) = (x^3-1)^{100}$

$$f'(x) = 100(x^3-1)^{99} \cdot 3x^2 = 300x^2(x^3-1)^{99}$$

3.4 - Parametric Form of 1st and 2nd Derivatives

If we substitute $y = g(t)$ and $x = f(t)$ in the equation $y = F(x)$, we get

$$g(t) = f\big(f(t)\big)$$

and so, if g, F, and f are differentiable, the chain rule gives

$$g'(t) = F'\big(f(t)\big)f'(t) = F'(x)f'(t)$$

If $f'(t) \neq 0$, we can solve for $F'(x)$:

$$F'(x) = \frac{g'(t)}{f'(t)}$$

Since the slope of the tangent to the curve $y = F(x)$ at $\big(x, F(x)\big)$ is $F'(x)$, this enables us to find tangents to parametric curves without having to eliminate the parameter.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad \text{if} \quad \frac{dx}{dt} \neq 0$$

It can be seen from this equation that the curve has a horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$) and it has a vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$).

The second derivative can be found by replacing y by $\frac{dy}{dx}$:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Example:

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the cycloid $x = r(\theta - \sin\theta), y = r(1 - \cos\theta)$.

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{r\sin\theta}{r(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$$

In order to find $\frac{d^2y}{dx^2}$, we first compute

$$\frac{d}{d\theta}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{\sin\theta}{1-\cos\theta}\right) = \frac{\cos\theta\,(1-\cos\theta) - \sin\theta\sin\theta}{(1-\cos\theta)^2} = \frac{\cos\theta - 1}{(1-\cos\theta)^2}$$
$$= -\frac{1}{1 - cos\,\theta}$$

Then

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\frac{dx}{d\theta}} = \frac{-\frac{1}{1-\cos\theta}}{r(1-\cos\theta)} = -\frac{1}{r(1-\cos\theta)^2}$$

3.5 - Slope in Polar Form

To find the tangent line to a polar curve $r = f(\theta)$ we regard θ as a parameter and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad\qquad y = r\sin\theta = f(\theta)\sin\theta$$

The, using the method for finding slopes of parametric curves, we have

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We find horizontal tangents by finding the points where $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$), and vertical tangents where $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$). If we are looking for tangents at the pole, then $r = 0$ and this equation simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \qquad \frac{dr}{d\theta} \neq 0$$

Example:

Find $\frac{dy}{dx}$ for the cardiod $r = 1 + \sin\theta$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{\cos\theta\sin\theta + (1+\sin\theta)\cos\theta}{\cos\theta\cos\theta - (1+\sin\theta)\sin\theta} = \frac{\cos\theta\,(1 + 2\sin\theta)}{1 - 2\sin^2\theta - \sin\theta}$$

$$= \frac{cos\,\theta\,(1 + 2\,sin\,\theta)}{(1 + sin\,\theta)(1 - 2\sin\theta)}$$

3.6 - Implicit Differentiation and Related Rates Strategy

Implicit differentiation

When given a function involving y and x, finding y in terms of x in order to take the derivative in not necessary because the method of **implicit differentiation** can be used. This consists of differentiating both sides with respect to x, and then solving the resulting equation for y' or $\frac{dy}{dx}$.

Example:

If $x^2 + y^2 = 25$, find $\frac{dy}{dx}$.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x, and using the chain rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus

$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Related rates strategy

If we are pumping air into a balloon, both the volume and the radius of the balloon are increasing and their rates of increas are related to each other. But it is much easier to measure directly the rate of increase of the volume than the rate on increase of the radius.
In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the chaine rule to differentiate both sides with respect to time.

Example:

Air is being pumped into a balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diamter is 50 cm?

We start by indentifying two things:

the *given infomation*:

the rate of increase of the volume of air is 100 cm^3/s and the *unknown*:

the rate of increase of the radius when the diameter is 50 cm

In order to express these quantities mathmatically we introduce some suggestive *notation*:

Let V be the volume of the balloon and let r be its radius.

The key thing to remember is that rates of change are derivatives. In this problem, the volume and the radius are both functions of the time t. The rate of increase of the volume with respect to time is the derivative $\frac{dV}{dt}$ and the rate of increase of the radius is $\frac{dr}{dt}$. We can therefore restate the given and the unknown as follows:

Given: $\frac{dV}{dt} = 100$ cm^3/s

Unknown: $\frac{dr}{dt}$ when $r = 25$ cm

We must relate V and r by the formula for the volume of a sphere:

$$V = \frac{4}{3}\pi r^3$$

Then differentiate both sides

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$$

and solve for the unknown quantity:

$$\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}$$

If we put $r = 25$ and $\frac{dV}{dt} = 100$ in this equation, we obtain

$$\frac{dr}{dt} = \frac{1}{4\pi(25)^2}100 = \frac{1}{25\pi}$$

The radius of the balloon is increasing at the rate of $\frac{1}{25\pi}$ cm/s

3.7 - Logarithmic Differentiation

The calculation of derivatives of complicated functions involving products, quotients, or powers can often be simplified by taking logarithms. This is called **logarithmic differentiation**.

Steps in logarithmic differentiation:

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the laws of logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y' or $\frac{dy}{dx}$.

If $f(x) < 0$ for some values of x, then $\ln f(x)$ is not defined, but we can write $|y| = |f(x)|$

Example:

Prove the general version of the power rule.

Proof

Let $y = x^n$ and use logarithmic differentiation:

$$\ln|y| = \ln|x|^n = n\ln|x| \qquad x \neq 0$$

Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

Hence

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

3.8 - Equation of a Normal Line

The **normal line** to a curve C at point P is the line that passes through P and is perpendicular to the tangent line of C at P.

Example:

Find the equation of a normal line to the parabola $y = 1 - x^2$ at the point $(2, -3)$.

First, we find the slope of y at $(2, -3)$:

$\frac{dy}{dx} = -2x$ at $(2, -3)$ $\frac{dy}{dx} = -2(2) = -4$

perpendicular slope $= \frac{1}{4}$

So, the normal line equation is:

$$y + 3 = \frac{1}{4}(x - 2)$$

3.9 - Instantaneous Rate of Change vs. Average Rate of Change

Suppose y is a quantity that depends on another quantity x. Thus, y is a function of s and we write $y = f(x)$. If x changes from x_1 to x_2, the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

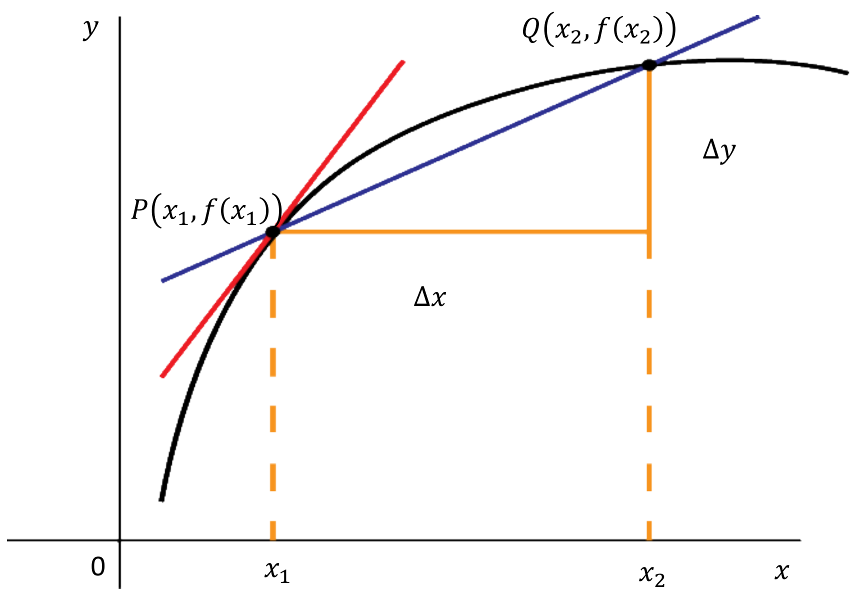
and the corresponding change in y is:

$$\Delta y = f(x_2) - f(x_1)$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the **average rate of change of y with respect to x** over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in the figure below.



average rate of change $= m_{PQ}$
instantaneous rate of change $=$ slope of tangent at P

We consider the average rate of change over smaller and smaller intervals by letting $x_2 \to x_1$ and therefore letting $\Delta x \to 0$.The limit of these average rates is called the **instantaneous rate of change of y with respect to x** at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$at $P(x_1, f(x_1))$.

$$\text{instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example:

Temperature readings T (in degrees Celsius) were recorded every hour starting at midnight. The time x is measured in hours from midnight.

(a) Find the average rate of change with respect to time from noon to 1 P.M.

(b) Estimate the instantaneous rate of change at noon.

(a) $\dfrac{\Delta T}{\Delta x} = \dfrac{T(13) - T(12)}{13 - 12} = \dfrac{16.0 - 14.3}{1} = 1.7℃/\mathrm{h}$

(b) We use the data given to sketch a smooth curve that approximates the graph of the temperature functions, then draw the tangent at point P where $x = 12$. Then, we get

$$\frac{dT}{dx} = \frac{10.3}{5.5} \approx 1.9$$

Therefore, the instantaneous rate of change of temperature with respect to time at noon is about 1.9℃/h

3.10 - Relationship Between the Increasing and Decreasing Behavior and Concavity of f and the Signs of f' and f''

Increasing/decreasing test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval
(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval

Proof

Let $f(x)$be a differentiable function defined on $[a, b]$. Let x_1 and x_2 be any two numbers in the interval with $x_1 < x_2$. We have to show that $f(x_1) < f(x_2)$ (given that $f'(x) > 0$. By the mean value theorem, there is a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

Now $f'(c) > 0$ by assumption and $x_2 - x_1 > 0$ because $x_1 < x_2$. Thus, the right side of the equation is positive, so

$$f(x_2) - f(x_1) > 0 \qquad \text{or} \qquad f(x_1) < f(x_2)$$

Showing that f is increasing.

Concavity

If the graph of f lies above all tangents on the interval I, then it is called **concave upward** on I. If the graph of f lies below all tangents on I, then it is called **concave upward** on I.

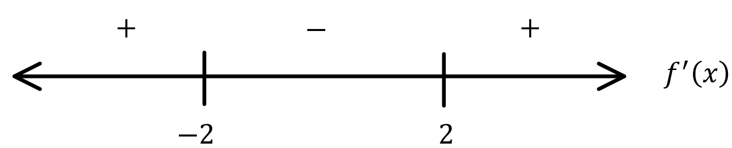
Concavity test

(a) If $f''(x) > 0$ for all x in I, the graph of f is concave up on I
(b) If $f''(x) < 0$ for all x in I, the graph of f is concave down on I

Example:

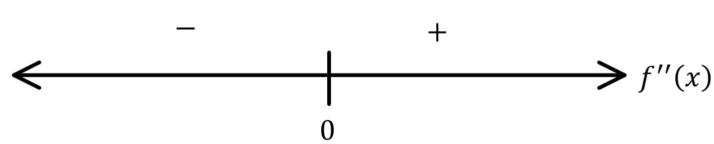
Find the intervals for increasing/decreasing, and for concavity of $f(x) = x^3 - 12x + 1$

$f'(x) = 3x^2 - 12 = 0$
$3(x^2 + 4) = 0$
$3(x + 2)(x - 2) = 0$
$x = \pm 2$



$f(x)$ is increasing on $(-\infty, -2)$ and $(2, \infty)$
$f(x)$ is decreasing on $(-2,2)$

$f''(x) = 6x = 0$
$x = 0$



$f''(x) > 0$ on $(-\infty, 0)$, therefore, the graph of $f(x)$ is concave down on $(-\infty, 0)$
$f''(x) > 0$ on $(0, \infty)$, therefore, the graph of $f(x)$ is concave up on $(0, \infty)$

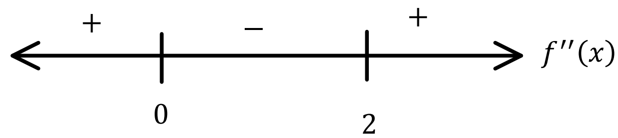
3.11 - Points of Inflection

A point P on a curve is called a point of inflection if the curve changes from concave upward to concave downward, or vise versa, at P

Example:

Find the x values for the points of inflection for the graph of $f(x) = x^4 - 4x^3$.

$f'(x) = 4x^3 - 12x^2$
$f''(x) = 12x^2 - 24x = 0$
$12x(x-2) = 0$
$x = 0, 2$



There is an inflection point where $x = 0$ because $f''(x)$ changes from positive to negative so the graph of $f(x)$ changes from concave up to concave down, and there is an inflection point where $x = 2$ because $f''(x)$ changes from negative to positive so the graph of $f(x)$ changes from concave down to concave up.

3.12 - Mean Value Theorem

Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

   Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

   or, equivalently,

$$f(a) - f(b) = f'(c)(b - a)$$

Example:

$f(x) = x^3 - x$, find c on $[0,2]$

1. f is continuous on $[0,2]$ (f is a ploynomial)
2. f is differentiable on $(0,2)$

$$f'(x) = 3x^2 - 1$$

Then, by the mean value theorem, there exists a number c in $(0,2)$ such that

$$f(2) - f(0) = f'(c)(2 - 0)$$
$$6 - 0 = (3c^2 - 1)(2)$$
$$6 = 6c^2 - 2$$
$$8 = 6c^2$$
$$c^2 = \frac{4}{3}$$
$$c = \pm\frac{2}{\sqrt{3}}$$

as c can only lie in $(0,2)$, $c = \frac{2}{\sqrt{3}}$ is the only valid answer

3.13 - Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).
3. $f(a) = f(b)$

Then there is a number c in (a,b) such that $f'(c) = 0$.

Proof

There are three cases:

Case 1: **$f(x) = k$, a constant**

Then $f'(x) = 0$, so the number c can be taken to be any number in (a,b).

Case 2: **$f(x) > f(a)$ for some x in (a,b)**

By the extreme value theorem, f has a maximum value somewhere in $[a,b]$. Since $f(a) = f(b)$, it must attain this maximum value at a number c in the open interval (a,b). Then f has a local maximum at c and, by hypothesis 2, f is differentiable at c. Therefore, $f'(c) = 0$ by Fermat's theorem.

Case 3: **$f(X) < f(a)$ for some x in (a,b)**

By the extreme value theorem, f has a minimum value in $[a,b]$ and, since $f(a) = f(b)$, it attains this minimum value at a number c in (a,b). Again $f'(c) = 0$ by Fermat's theorem.

Example:

Show that there is a number c on $[0,4]$ for $f(x) = x^2 - 4x + 1$.

1. $f(x)$ is continuous on $[0,4]$ (polynomial function)
2. $f(x)$ is differentiable on $[0,4]$
3. $f(0) = 1$ and $f(4) = 1$, so $f(0) = f(4)$

Therefore, by Rolle's theorem, there is a number c on $[0,4]$ such that $f'(c) = 0$

3.14 - 1st and 2nd Derivative Tests

First derivative test:

Suppose c is a critical number of a continuous function f.

a) If f' changes from positive to negative at c, then f has a local maximum at c.
b) If f' changes from negative to positive at c, then f has a local minimum at c.
c) If f' does not change sign (that is, f' is positive on both side of c or negative on both sides), then f has no local maximum of minimum at c.

Second derivative test:

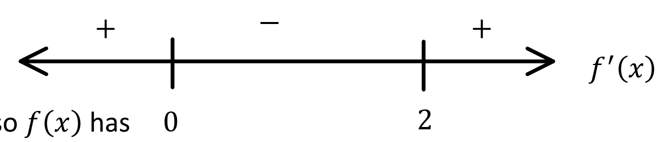
Suppose f'' is continuous near c.

a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.

Example:

Find the local maximums and/or minimums of $f(x) = 5 - 3x^2 + x^3$ with both the 1st and 2nd derivative tests.

$$f'(x) = -6x + 3x^2 = 3x^2 - 6x = 0$$
$$3x(x-2) = 0$$
$$x = 0, 2$$



f' changes from positive to negative at $x = 0$, so $f(x)$ has a local maximum at $x = 0$.
f' changes from negative to positive at $x = 2$, so $f(x)$ has a local minimum at $x = 2$.

$$f''(x) = 6x - 6$$

Using the critical numbers $x = 0{,}2$ from $f'(x)$

$f''(0) = 6(0) - 6 = -6 < 0$, so $f(x)$ has a maximum at $x = 0$.
$f''(2) = 6(2) - 6 = 6 > 0$, so $f(x)$ has a minimum at $x = 0$.

3.15 - Steps to Finding Absolute Extrema on an Interval & Optimization Strategy

The closed interval method

To find absolute maximum and minimum values of a continuous function f on a closed interval $[a,b]$:

1. Find the values of f at the critical numbers of f in (a,b).
2. Find the values of f at the end points of the interval.
3. The largest values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Example:

$$f(x) = x^3 - 3x^2 + 1 \qquad -\frac{1}{2} \leq x \leq 4$$

Find the absolute extrema values

$$f(x) = x^3 - 3x^2 + 1$$
$$f'(x) = 3x^2 - 6x = 3x(x-2) = 0$$

$x = 0, 2$ both exist in the closed interval $\left[-\frac{1}{2}, 4\right]$.

x		$f(x)$
$-\frac{1}{2}$	endpoint	$\frac{1}{8}$
0	critical #	1
2	critical #	-3
4	endpoint	17

absolute minimum at $f(2) = -3$
absolute maximum at $f(4) = 17$

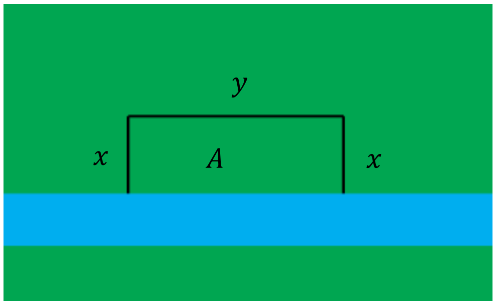
Optimization Strategy

Steps in solving optimization problems

1. **Understand the problem**: The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions?
2. **Draw a diagram**: In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce notation**: Assign a symbol to the quantity that is to be maximized or minimized (lets call it Q now). Also select symbols (a, b, c, $\cdots$, x, y) for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols - for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus, Q will be expressed as a function of one variable x, say, $Q = f(x)$. Write the domain of this function.
6. Find the absolute maximum or minimum value of f.

Example:

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area.



The figure to the right illustrates the general case. We wish to maximize the area A of the rectangle. Let x and y be the depth and width of the of the rectangle (in feet). Then we express A in terms of x and y:

$$A = xy$$

We want to express A as a function of just one variable, so we eliminate y by expressing it in terms of x. To do this we use the given information that the total length of the fencing is 2400 ft. Thus

$$2x + y = 2400$$
$$y = 2400 - 2x$$

Which gives

$$A = x(2400 - 2x) = 2400x - 2x^2$$

Note that $0 \leq x \leq 1200$ (otherwise $A < 0$). So the function we wish to maximize is

$$A(x) = 2400x - 2x^2 \qquad 0 \leq x \leq 1200$$

$$A'(x) = 2400 - 4x$$

x		$A(x)$
0	endpoint	0
600	critical #	720000
1200	endpoint	0

$$y = 2400 - 2(600)$$

$$y = 1200$$

So the dimensions of the field with the largest area is 600x1200.

3.16 - Horizontal and Vertical Tangents in Parametric and Polar Form

To find horizontal and vertical tangents of a parametric equation with $x = f(t)$ and $y = g(t)$, we look at the 1st derivative for parametric equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if} \qquad \frac{dx}{dt} \neq 0$$

It can be seen from this equation that the curve has a horizontal tangent when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$) and it has a vertical tangent when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$). In the event that both $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} = 0$, then l'Hospital's rule must be applied to simplify the function and obtain a limit. If the limit is euqal to 0, then it is a horizontal tangent. If the limit does not exist, it is a vertical tangent.

To find horizontal and vertical tangents of a polar equation with $r = f(\theta)$, we look at the 1st derivative for polar equations:

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

We find horizontal tangents by finding the points where $\frac{dy}{d\theta} = 0$ (provided that $\frac{dx}{d\theta} \neq 0$), and vertical tangents where $\frac{dx}{d\theta} = 0$ (provided that $\frac{dy}{d\theta} \neq 0$). If we are looking for tangents at the pole, then $r = 0$ and this equation simplifies to

$$\frac{dy}{dx} = \tan\theta \qquad \text{if} \qquad \frac{dr}{d\theta} \neq 0$$

Example:

$$x = t^3 - t, \qquad y = 2 - 5t$$

$$\frac{dx}{dt} = 3t^2 - 1 \qquad \frac{dy}{dt} = -5 \qquad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-5}{3t^2 - 1}$$

no horizontal tangents because $\frac{dy}{dt} = -5 \neq 0$

$$\frac{dx}{dt} = 3t^2 - 1 = 0$$

$$t^2 = \frac{1}{3} \qquad t = \pm\frac{1}{\sqrt{3}}$$

vertical tangents occur at $t = \frac{1}{\sqrt{3}}$ and $t = -\frac{1}{\sqrt{3}}$

3.17 - Relationship Between Position, Velocity, and Acceleration

A position function states the position of a particle at time t ($s(t) = f(t)$).
The derivative of the position function is the velocity function ($s'(t) = v(t)$).
The second derivative of the position function, or the derivative of the velocity function is the acceleration function ($s''(t) = v'(t) = a(t)$)

Example:

A particle moves along a vertical line so that its coordinate at time t is $y = t^3 - 12t + 3 \qquad t \geq 0$.

Velocity function (y')

$$v(t) = 3t^2 - 12$$

Acceleration function ($v'(t)$)

$$a(t) = 6t$$

3.18 - Velocity and Acceleration Vector

Suppose that a particle moves through space so that its position vector at time t is $r(t)$. For small values of h, the vector

$$\frac{r(t+h)-r(t)}{h}$$

approximates the direction of the particle moving along the curve $r(t)$. Its magnitude measures the size of the displacement vector per unit time. The vecotr gives the average velocity over a time interval of length h and its limit is the **velocity vector** $v(t)$at time t:

$$v(t)=\lim_{h\to 0}\frac{r(t+h)-r(t)}{h}=r'(t)$$

Thus the velocity vector is also the tangent vector and points in the direction of the tangent line.

As in the case of one-dimensional movement, the **acceleration vector** is defined as the derivative of the velocity vector:

$$a(t)=v'(t)=r''(t)$$

Example:

$r(t)=\langle t^3,t^2\rangle, \qquad t\geq 0$

$$v(t)=r'(t)=\langle 3t^2,2t\rangle$$

$$a(t)=v'(t)=\langle 6t,2\rangle$$

3.19 - Speed and How it is Calculated

The **speed** of a particle at time t is the magnitude of the velocity, that is, $|v(t)|$. This is appropriate because

$$|v(t)| = |r'(t)| = \frac{ds}{dt} = \text{rate of change of distance with respect to time}$$

Example:

$r(t) = \langle t^3, t^2 \rangle, \qquad t \geq 0$

Find the speed at $t = 4$.

$$v(t) = \langle 3t^2, 2t \rangle$$

$$|v(t)| = \sqrt{(3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 4t^2}$$

$t = 4$

$$\sqrt{9(4)^4 + 4(4)^2} \approx 48.6621$$

3.20 - Slope Fields

It is impossible to solve most differential equations, and even harder to graph them. Given the equation $y' = x + y$, it can be deduced that the slope at any point (x, y) on the graph is equal to the sum of the x and y coordinates of that point.

To find the slopes, or y', plug in the x and y coordinates for the given point. Then, at the point (x, y) graph the slope of the line at that specific point.

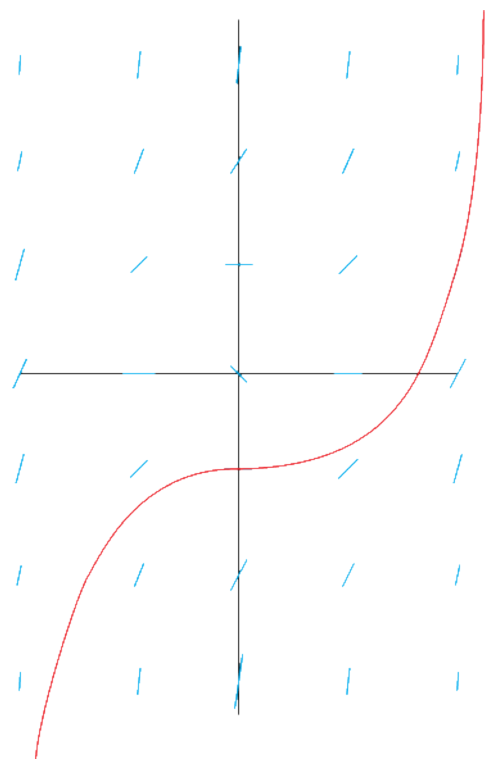
These line segments on a graph are called a **directional field** or a **slope field**.

Example:

Sketch the slope field for $y' = x^2 + y^2 - 1$ and the line with an initial value of $y(0) = -1$.

x	-2	-1	0	1	2	-2	-1	0	1	2
y	0	0	0	0	0	1	1	1	1	1
slope	3	0	-1	0	3	4	1	0	1	4

slope field approximation and solution curve for $y(0) = -1$



3.21 - Euler's method of Numerical Solutions

The idea of directional fields can be used to find numerical approximations to solutions of differential equations.
Though Euler's method does not give the exact solution to an initial-value problem, it gives an approximation using the general first-order initial-value problem.

For the general first-order initial-value problem $y' = F(x, y), y(x_0) = y_0$, our aim is to find approximate values for the solution at equally spaced number x_0, $x_1 = x_0 + hF(x, y)$, $x_2 = x_1 + hF(x, y)$, $\cdots$, where h is the step size. The differential equation tells us that the slope at (x_0, y_0) is $y' = F(x_0, y_0)$, so the approximate value of the solution when $x = x_1$ is

$$y_1 = y_0 + hF(x_0, y_0)$$

Similarly

$$y_2 = y_1 + F(x_1, y_1)$$

In general,

$$y_n = y_{n-1} + F(x_{n-1}, y_{n-1})$$

Smaller values for h (or Δx) give approximations closer to the actual value.

Example:

Estimate $y(0.3)$ for:

$$y' = x + y \qquad y(0) = 1 \qquad \Delta x = .1$$

$$y_1 = y_0 + \Delta x(x_0 + y_0) = 1 + 0.1(0 + 1) = 1.1$$

$$y_2 = y_1 + \Delta x(x_1 + x_2) = 1.1 + 0.1(.1 + 1.1) = 1.22$$

$$y + 3 = y_2 + \Delta x(x_2 + y_2) = 1.22 + 0.1(.2 + 1.22) = 1.362$$

so, $y(0.3) \approx 1.362$

3.22 - Local Linearization

The idea is that it might be easy to calculate a value $f(a)$ of a function, but difficult (or even impossible) to compute nearby values of f. So, we settle for the easily computed values of the linear function L whose graph is the tangent line of f at $(a, f(a))$. In other words, we use the tangent line at $(a, f(a))$ as an approximation to the curve $y = f(x)$ when x is near a. An equation of this tangent line is

$$y = f(a) + f'(a)(x - a)$$

and the approximation

$$f(x) \approx f(a) + f'(a)(x - a)$$

is called the **linear approximation** or **tangent line approximation** of f at a. The linear function whose graph is this tangent line, that is,

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** of f at a.

Example:

Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a = 1$ and use it to approximate the numbers $\sqrt{3.98}$ and $\sqrt{4.05}$.

$$f'(x) = \frac{1}{2}(x+3)^{\frac{-1}{2}} = \frac{1}{2\sqrt{x+3}}$$

$$f(1) = 2 \qquad\qquad f'(1) = \frac{1}{4}$$

$$L(x) = f(1) + f'(1)(x - 1) = 2 + \frac{1}{4}(x - 1) = \frac{7}{4} + \frac{x}{4}$$

$$\therefore \sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4}$$

So,

$$\sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995$$

and

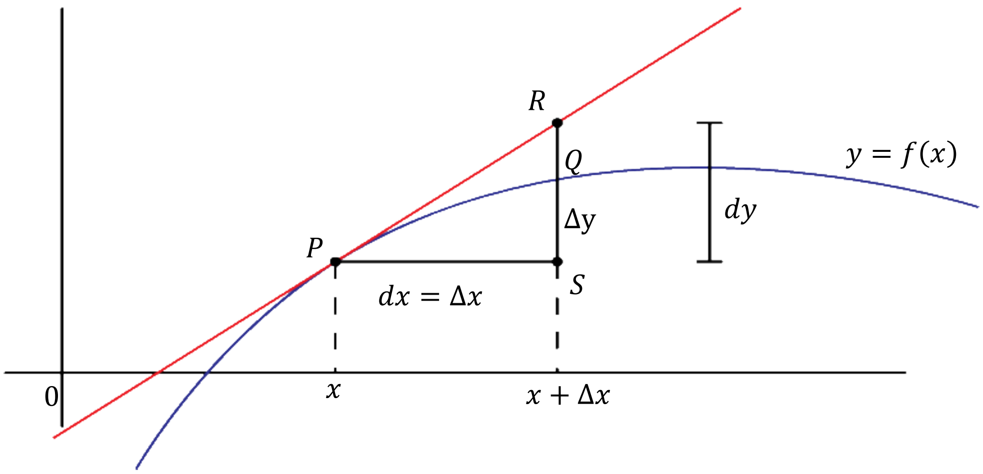
$$\sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125$$

3.23 - Definition of a Differential

If $y = f(x)$ where f is a differentiable function, then the **differential** dx is an independent variable; that is dx can be given the value of any real number. The **differential** dy is then defined in terms of dx by the equation

$$dy = f'(x)dx$$

Therefore, dy is a dependent variable; it depends on the value of x and dx.



The slope of the tangent line PR is the derivative $f'(x)$. Thus, the directed distance from S to R is $f'(x)dx = dy$. Therefore, dy represents the amount that the tangent line rises of falls (the change in the linearization), whereas Δy represents the amount that the curve $y = f(x)$ rises or falls when x changes by an amount dx.

Example:

Compare the values of Δy and dy if $y = f(x) = x^3 + x^2 - 2x + 1$ and x changes from 2 to 2.05.

$$f(2) = 2^3 + 2^2 - 2(2) + 1 = 9$$

$$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$$

$$\Delta y = f(2.05) - f(2) = 0.717625$$

$$dy = f'(x)dx = (3x^2 + 2x - 2)dx = [3(2)^2 + 2(2) - 2]0.05 = 0.7$$

4.1 - Riemann Sums

A summation taken to a limit is used to find the area under a curve.

The **area** A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} [f(x_1)\Delta\mathrm{x} + f(x_2)\Delta\mathrm{x} + \cdots + f(x_n)\Delta\mathrm{x}]$$

Since we are assuming that f is continuous, it can be prooved that the limit always exists. It can also be shown that we get the same value if we use left endpoints:

$$A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} [f(x_0)\Delta\mathrm{x} + f(x_1)\Delta\mathrm{x} + \cdots + f(x_{n-1})\Delta\mathrm{x}]$$

Or we could take the height of the ith rectangle to be the value of f at any number x_i^* in the ith subinterval $[x_{i-1}, x_i]$. We call the numbers x_1^*, x_2^*, $\cdots$, x_n^* the **sample points**. So a more general exression for the area of S is

$$A = \lim_{n\to\infty} [f(x_1^*)\Delta\mathrm{x} + f(x_2^*)\Delta\mathrm{x} + \cdots + f(x_n^*)\Delta\mathrm{x}]$$

Which can be writen in **sigma notation** as:

$$\sum_{i=1}^{n} f(x_i)\Delta\mathrm{x} = f(x_1)\Delta\mathrm{x} + f(x_2)\Delta\mathrm{x} + \cdots + f(x_n)\Delta\mathrm{x}$$

which is called a Reimann sum. Sample points can be chosen from the left, right, or midpoints.

Example:

Given the following table of a function, find the sum using left, right, and midpoints.

x	5	10	15	20	25	30	35
$f(x)$	100	185	319	447	548	700	825

The left endpoints are: 5, 10, 15, 20, 25, 30. So

$$L_5 = 100(5) + 185(5) + 319(5) + 447(5) + 548(5) + 700(5) = 11495$$

The right endpoints are 10, 15, 20, 25, 30, 35. So

$$R_5 = 185(5) + 319(5) + 447(5) + 548(5) + 700(5) + 825(5) = 15120$$

For midpoints, we choose every other point (10, 20, 30). So

$$M_3 = 185(10) + 447(10) + 700(10) = 13320$$

4.2 - Summation Formulas

When evaluating a definite integral, here are some sums to help the evaluation and simplification process.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

The remaining rules are simple rules for working with sigma notation:

$$\sum_{i=1}^{n} c = nc$$

$$\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} (a_i - b_i) = \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i$$

Example:

Find the Reimann sum of $f(x) = x^3 - 6x$ over $[0,3]$.

$$\Delta x = \frac{b-a}{n} = \frac{3}{n} \qquad x_i = \frac{3i}{n}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta \mathrm{x} = \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\left(\frac{3i}{n}\right)^3 - 6\left(\frac{3i}{n}\right)\right] \lim_{n\to\infty} \frac{3}{n}\sum_{i=1}^{n}\left[\frac{27}{n^3}i^3 - \frac{18}{n}i\right]$$

$$= \lim_{n\to\infty}\left[\frac{81}{n^4}\sum_{i=1}^{n} i^3 - \frac{54}{n^2}\sum_{i=1}^{n} i\right] = \lim_{n\to\infty}\left\{\frac{81}{n^4}\left[\frac{n(n+1)}{2}\right]^2 - \frac{54}{n}\frac{n(n+1)}{2}\right\}$$

$$= \lim_{n\to\infty}\left[\frac{81}{4}\left(1+\frac{1}{n}\right)^2 - 27\left(1+\frac{1}{n}\right)\right] = \frac{81}{4} - 27 = -6.75$$

4.3 - Trapezoidal Rule

$$\int_a^b f(x)dx \approx T_n = \frac{\Delta x}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

where $\Delta x = \frac{b-a}{n}$ and $x_1 = a + i\Delta x$.

Error bound for Trapezoidal Rule and Midpoint Rule:

Suppose $K = \max|f''(x)|$ on $[a, b]$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad \text{and} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

Example:

Use the Trapezoidal Rule to approximate $\int_1^2 \left(\frac{1}{x}\right) dx$ with $n = 5$.

$$a = 1 \qquad b = 2 \qquad \Delta x = \frac{2-1}{5} = .2$$

$$T_5 = \frac{0.2}{2}[f(1) + 2f(1.2) + 2f(1.4) + 2f(1.6) + 2f(1.8) + f(2)]$$

$$= \frac{1}{10}\left(\frac{1}{1} + \frac{2}{1.2} + \frac{2}{1.4} + \frac{2}{1.6} + \frac{2}{1.8} + \frac{1}{2}\right) = 0.695634$$

4.4 - Basic Properties of Definite Integrals

Definition of a definite limit

f is a continuous function defined for $a \leq x \leq b$. We divide the interval $[a, b]$ into n subintervals of equal width ($\Delta x = \frac{b-a}{n}$). We let x_0 (which equals a), x_1, x_2, $\cdots$, x_n(which equals b) be the endpoints of the subintervals. Sample points are chosen so that x_i lies in the ith subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x)\Delta\mathrm{x}$$

Basic properties

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^a f(x)dx = 0$$

Assuming f and g are continuous functions and c is any constant:

$$\int_a^b cdx = c(b-a)$$

$$\int_a^b cf(x)dx = c\int_a^b f(x)dx$$

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$\int_a^b [f(x)-g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Example:

Evaluate $\int_0^1 (4+3x^2)dx$.

$$\int_0^1 (4+3x^2)dx = \int_0^1 4dx + 3\int_0^1 x^2 dx$$

$$\int_0^1 4 = 4(1-0) = 4$$

$$3\int_0^1 x^2 = 3\left(\frac{1}{3}x^3\right)\Big]_0^1 = 3\left(\frac{1}{3}\right) = 1$$

So

$$\int_0^1 (4+3x^2)dx = \int_0^1 4dx + 3\int_0^1 x^2 dx = 4+1 = 5$$

4.5 - Fundamental Theorem of Calculus (Part 1)

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \leq x \leq b$$

is continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) = f(x)$.

Example:

Find the derivative of the function $g(x) = \int_0^x \sqrt{1 + t^2}dt$.

Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + x^2}$$

4.6 - Fundamental Theorem of Calculus (Part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(a) - F(b)$$

where $F$is any antiderivative of f, that is, a function such that $F' = f$.

Example:

Evaluate $\int_1^3 e^x dx$.

$$\int e^x = e^x + C = F(x)$$

$$\int_1^3 e^x dx = F(3) - F(1) = e^3 - e$$

4.7 - Integration Rules

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad (n \neq 0)$$

$$\int \frac{1}{x} dx = \ln x + C$$

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \cot x \, dx = \ln|\sin x| + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}} dx = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

Example:

(a) $\int x^3 dx = \frac{x^4}{4} + C$

(b) $\int 4^x dx = \frac{4^x}{\ln 4} + C$

(c) $\int \frac{1}{x^2 + 25} dx = \frac{1}{5} \tan^{-1}\left(\frac{x}{5}\right) + C$

(d) $\int \frac{1}{\sqrt{36 - x^2}} dx = \sin^{-1}\left(\frac{x}{6}\right) + C$

4.8 - Integration by Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Note: For definite integrals, the limits must be converted to u-limits.

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example:

Find $\int x^3 \cos(x^4 + 2)\, dx$.

$$\int x^3 \cos(x^4 + 2)\, dx = \frac{1}{4}\int \cos u\, du$$

$$u = x^4 + 2$$
$$du = 4x^3 dx$$

$$= \frac{1}{4}\sin(x^4 + 2) + C$$

4.9 - Integration by Parts

The product rule states that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In notation of indefinite integrals:

$$\int [f(x)g'(x) + g(f'(x)]dx = f(x)g(x)$$

This can be rearranged to:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Example:

1. Find $\int x \sin x \, dx$.

$$\int x \sin x \, dx = -x \cos x - \int -\cos x \, dx$$

$$u = x \qquad dv = \sin x \, dx$$
$$du = dx \qquad v = -\cos x$$

$$= -x \cos x + \sin x + C$$

2. Find $\int \ln x \, dx$.

$$u = \ln x \qquad dv = dx$$
$$du = \frac{1}{x} \qquad v = x$$

$$\int \ln x \, dx = x \ln x - \int x \frac{1}{x} dx = x \ln x - x + C$$

3. Find $\int x^2 e^x dx$.

Sign	$u = x^2$ du	$dv = e^x$ v
+	x^2	e^x
−	$2x$	e^x
+	2	e^x
−	0	e^x

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + C$$

4.10 - Integration of Simple Partial Fractions

To integrate any rational function (a ratio of two polynomials) by expressing it as a sum of simpler fractions, called **partial fractions**, that we already know how to integrate.

1. If the degree of the numerator is higher than that of the denominator, then perform long division.
2. Factor the denominator.
3. Split the rational function $\frac{P(x)}{Q(x)}$ into a sum of partial fractions in the form

$$\frac{A}{(ax+b)^i} \qquad \text{or} \qquad \frac{Ax+B}{(ax^2+bx+c)^i}$$

Case 1: The denominator $Q(x)$ is a product of distinct linear factors.
Case 2: $Q(x)$ is a product of linear factors, some of which are repeated.
Case 3: $Q(x)$ contains irreducible quadratic factors, none of which are repeated.
Case 4: $Q(x)$ contains a repeated irreducible quadratic factor.

Example:

$$\int \frac{1}{x^2+x-2}\,dx = \int \frac{A}{x+2} + \frac{B}{x-1}\,dx$$

x	$A(x-1)+B(x+2)$	1
1	$0+3B$	$B=\frac{1}{3}$
-2	$-3A+0$	$A=-\frac{1}{3}$

$$= \int \left[\frac{1}{3}\left(\frac{1}{x-1}\right) - \frac{1}{3}\left(\frac{1}{x+2}\right)\right] dx = \frac{1}{3}\ln|x-1| - \frac{1}{3}\ln|x+2| + C$$

$$= \ln \sqrt[3]{\left|\frac{x-1}{x+2}\right|} + C$$

4.11 - Improper Integrals

An integral is called **convergent** if its corresponding limit exists as a finite number and **divergent** if it does not exist.

Type 1: Infinite Integrals

(a) If $\int_a^t f(x)dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx$$

provided that this limit converges.

(b) If $\int_t^b f(x)dx$ exists for every number $t \leq b$, then

$$\int_\infty^b f(x)dx = \lim_{t\to\infty} \int_t^b f(x)dx$$

provided that this limit converges.

(c) If both $\int_a^\infty f(x)dx$ and $\int_{-\infty}^a f(x)dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^a f(x)dx + \int_a^\infty f(x)dx$$

In part (c) and real number a can be used.

Example:

Is the following limit convergent or divergent?

$$\int_1^\infty \frac{1}{x}dx$$

$$\lim_{b\to\infty} \int_1^b \frac{1}{x}dx = \lim_{b\to\infty} \ln|x|]_1^b = \lim_{b\to\infty} \ln b - \lim_{b\to\infty} \ln 1 = \infty$$

The integral diverges because the limit $= \infty$

Type 2: Discontinuous Integrands

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{t\to b^-} \int_a^t f(x)dx$$

if this limit converges.

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{t \to a^+} \int_t^b f(x)dx$$

if this limit converges.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)dx$ and $\int_c^b f(x)dx$ are convergent, then we define

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

Example:

Find $\int_2^5 \frac{1}{\sqrt{x-2}} dx$.

$$= \lim_{t \to 2^+} \int_t^5 \frac{1}{x-2} dx = \lim_{t \to 2^+} 2\sqrt{x-2}\,\Big]_t^5 = \lim_{t \to 2^+} 2\left(\sqrt{3} - \sqrt{t-2}\right) = 2\sqrt{3}$$

This integral converges.

Find $\int_0^3 \frac{1}{x-1} dx$.

$$= \int_0^1 \frac{1}{x-1} dx + \int_1^3 \frac{1}{x-1} dx$$

$$\int_0^1 \frac{1}{x-1} dx = \lim_{t \to 1^-} \int_0^t \frac{1}{x-1} dx = \lim_{t \to 1^-} \ln|x-1|\,\Big]_0^t = \lim_{t \to 1^-} \ln|t-1| - \ln|-1|$$
$$= \lim_{t \to 1^-} \ln|t-1| = -\infty$$

This integral diverges.

4.12 - Integration of Powers of sin and cos

Strategy for evaluating $\int \sin^m x \cos^n x\, dx$:

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x\, (\cos^2 x)^k \cos x\, dx$$

$$= \int \sin^m x\, (1 - \sin^2 x)^k \cos x\, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cos^m x \sin x\, dx$$

$$= \int (1 - \cos^2 x)^k \cos^{\mathrm{n}} x \sin x\, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \frac{1}{2}\sin 2x$$

Example:

$$\int \sin^5 x \cos^2 x\, dx = \int \sin^4 x \cos^2 x \sin x\, dx = \int (1 - \cos^2 x)^2 \cos^2 x \sin x\, dx$$

$$u = \cos x$$

$$= -\int (1 - u^2)^2 u^2 du = -\int (u^2 - 2u^4 + u^6) du$$

$$du = -\sin x\, dx$$

$$= -\left(\frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7}\right) + C = -\frac{\cos^3 x}{3} + \frac{2\cos^5 x}{5} - \frac{\cos^7 x}{7} + C$$

$$\int \sin^4 x\, dx = \int (\sin^2 x)^2 dx = \int \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) dx$$

$$= \frac{1}{4}\int \left[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)\right] dx$$

$$= \frac{1}{4}\int \left(\frac{3}{2} - 2\cos 2x + \frac{1}{2}\cos 4x\right) dx$$

$$= \frac{1}{4}\left(\frac{3}{2}\mathrm{x} - \sin 2x + \frac{1}{8}\sin 4x\right) + C$$

$$\int \cos^3 x \, dx = \int \cos^2 x \cos x \, dx = \int (1 - \sin^2 x) \cos x \, dx$$

$$= \int (1 - u^2) du = u - \frac{u^3}{3} + C$$

$u = \sin x$

$du = \cos x \, dx$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

4.13 - Finding Area of a Region

We can use the idea of the Riemann sum which finds the area under a curve to find the area between curves:

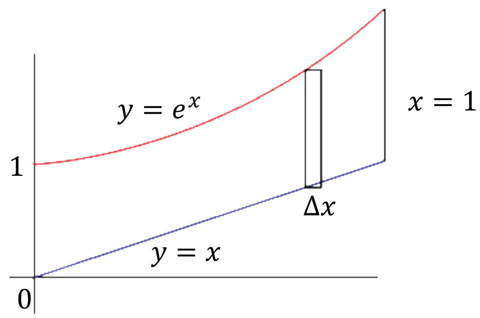
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) - g(x_i^*)]$$

We recognize this limit as the definite integral of $f - g$, so the area A of the region bounded by the curves $y = f(x)$ and $y = g(x)$, and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$ is

$$A = \int_a^b [f(x) - g(x)]dx$$

Example:

Find the area of the region bounded above by $y = e^x$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$.



$$A = \int_0^1 (e^x - x)dx = e^x - \frac{1}{2}x^2\Big]_0^1$$

$$= e - \frac{1}{2} - 1 = e - 1.5$$

4.14 - Area in Polar Coordinates

To find the area of a region bounded by a polar equation, we need to use the formula for the area of a sector of a circle

$$A = \frac{1}{2}r^2\theta$$

An Riemann sum for the total area bounded by a polar curve is

$$A = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}[f(\theta_i{}^*)]^2\Delta\theta$$

Therefore, the formula for the area A of a polar region $\mathcal{R}$ is

$$A = \int_a^b \frac{1}{2}[f(\theta)]^2 d\theta$$

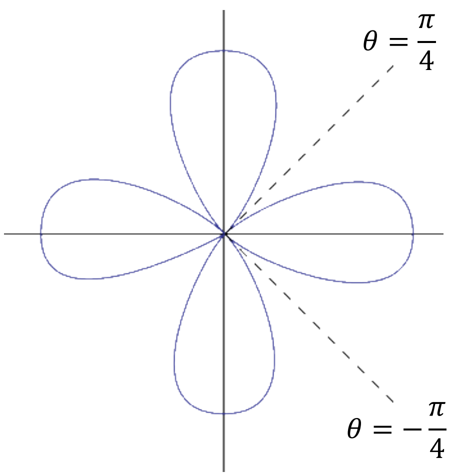
Which is often writen as

$$A = \int_a^b \frac{1}{2}r^2 d\theta$$

with the underdtanding that $r = f(\theta)$.

Example:

Find the area inclosed by one loop of the four-leaved rose $r = \cos 2\theta$.



$$A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}r^2 d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 2\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 4\theta)d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\frac{\pi}{4}} = \frac{\pi}{8}$$

4.15 - Finding the Average Value of a Function

It is easy to calculate the average value of finitely many numbers $y_1, y_2, \cdots, y_n$:

$$y_{\text{ave}} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

The same idea can be applied when finding the average value of a function. We choose sample points in the function and use the formula:

$$f_{\text{ave}} = \frac{f(x_1^*) + \cdots + f(x_n^*)}{n}$$

Since $\Delta x = \frac{b-a}{n}$, we can write $n = \frac{b-a}{\Delta x}$ and the average value becomes

$$\frac{f(x_1^*) + \cdots + f(x_n^*)}{\frac{b-a}{\Delta x}} = \frac{1}{b-a}[f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x]$$

$$= \frac{1}{b-a}\sum_{i=1}^{n} f(x_i^*)\Delta \mathrm{x}$$

Therefore, we define the average value of a function as

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx$$

Example:

Find the average value of the function $f(x) = 1 + x^2$ on the interval $[-1,2]$.

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{2-(-1)}\int_{-1}^{2}(1+x^2)dx$$

$$= \frac{1}{3}\left[x + \frac{x^3}{3}\right]_{-1}^{2} = 2$$

4.16 - Finding Volumes by Known Cross Section

We start by intersecting S with a plane and obtaining a plane region that is called a **cross-section** of S.

Definition of volume

Let S be a solid that likes between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P_x, through x and perpendicular to the x-axis, is $A(x)$, where A is a continuous function, then the **volume** of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta \mathrm{x} = \int_a^b A(x)dx$$

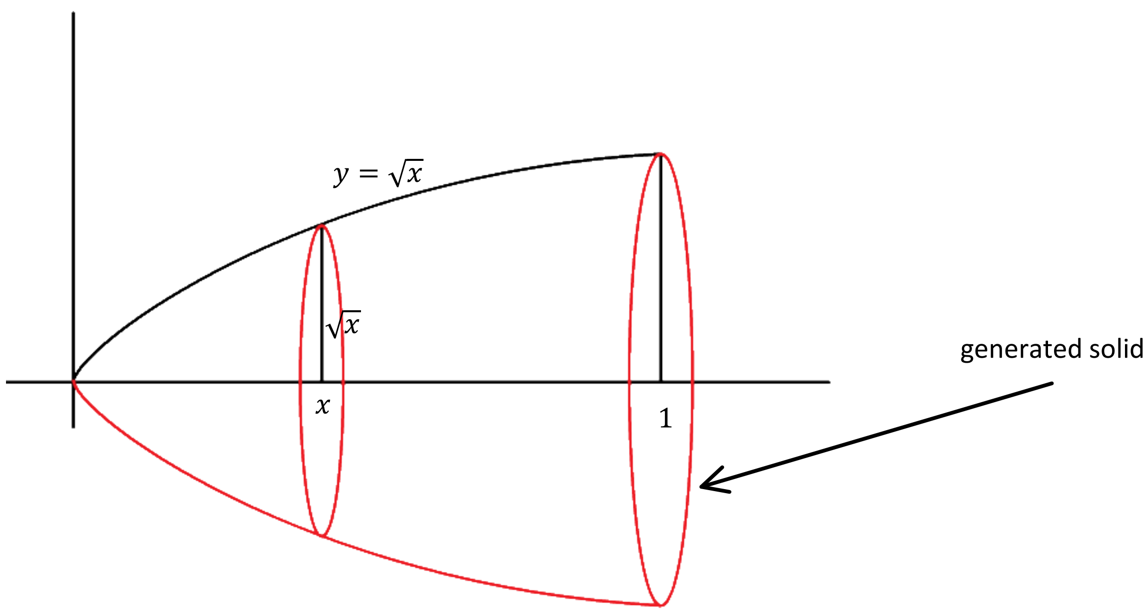
Example:

$y = \sqrt{x}$ from 0 to 1

$$A(x) = \pi(\sqrt{x})^2 = \pi x$$

The thickness of the approximating cylinder is Δx. Therefore, the volume is

$$V = \int_0^1 A(x)dx = \int_0^1 \pi x dx = \pi \frac{x^2}{2}\bigg]_0^1 = \frac{\pi}{2}$$



4.17 - Finding Volumes by Disc/Washer Method

Disc Method:

Uses the cross-section theorem with circular cross-sections:

Horizontal axis

$$V = \int_a^b \pi[f(x)^2]dx$$

Vertical axis

$$V = \int_c^d \pi[g(y)^2]dx$$

Washer Method:

Modified version of the disc method to use when there is a hole, a space between the axis, in the generated solid.

Horizontal axis

$$V = \int_a^b \pi(R^2 - r^2)dx$$

Vertical axis

$$V = \int_c^d \pi(R^2 - r^2)dx$$

Where R is the outer function and r is the inner function.

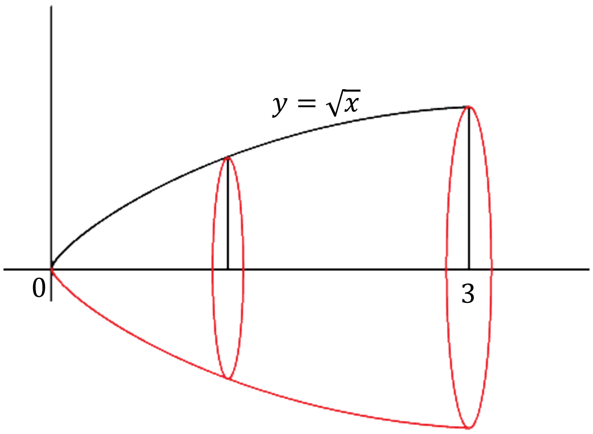
Example:

Find the volume of the solid generated by revolving $y = \sqrt{x}$ about the x-axis and about $y = 3$ from 0 to 9.

About the x-axis:

Disc method

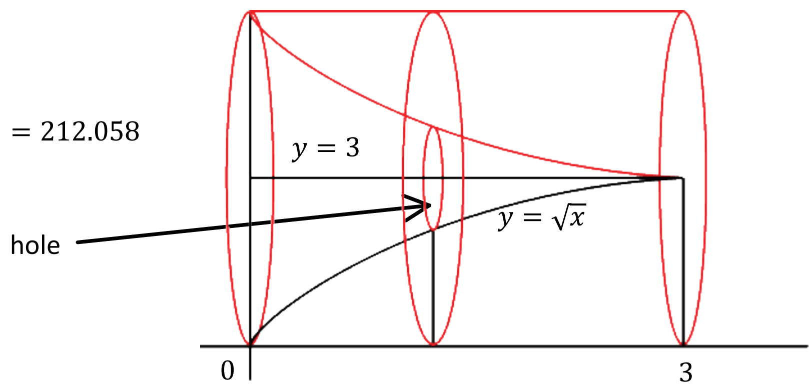
$$V = \int_a^b \pi(\sqrt{x})^2 dx = \int_a^b \pi x dx = \pi \frac{x^2}{2}\Big]_0^9 = \frac{81}{2}\pi$$



About the line $y = 3$:

Washer method

$$V = \int_a^b \pi(3^2 - (3 - \sqrt{x})^2)dx \ = 212.058$$



4.18 - Finding Volumes by the Shell Method

A shell is a hollow cylinder. Instead of making slices, the volume of a solid can be found by finding the sum of a group of shells.

Let r_0 = the radius of the outside function
r_1 = the radius of the inside function

So

$$V = \pi r_0^2 h - \pi r_1^2 h$$

$$= \pi h(r_0^2 - r_1^2)$$

$$= \pi h(r_0 + r_1)(r_0 - r_1)$$

$$= 2\pi h \Delta r \left(\frac{r_0 + r_1}{2}\right)$$

Sample point

Therefore, the volume of a solid generated by revolving the curve $y = f(x)$ about the y-axis from a to b is

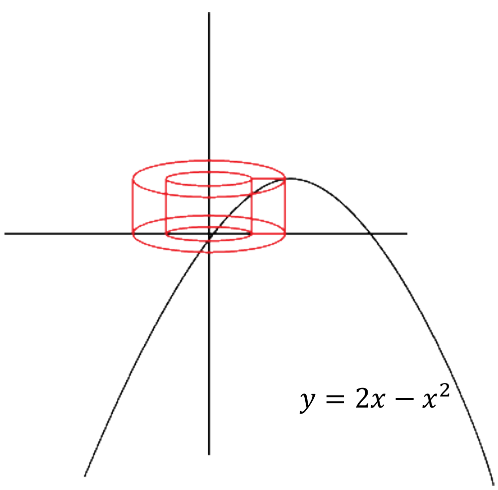
$$V = \int_a^b 2\pi r f(x) dx, \qquad 0 \le a < b$$

Revolving the curve $x = g(y)$ about the x-axis

$$V = \int_c^d 2\pi r g(y) dy, \qquad 0 \le c < d$$

Example:

Find the volume of the solid generated by revolving the region bounded by $y = 2x - x^2$ and $y = 0$ about the y-axis.



Find the intersection:

$2x - x^2 = 0$

$x(2 - x) = 0$

$x = 0, 2$

$$V = 2\pi \int_0^2 x(2x - x^2) dx = 8.378$$

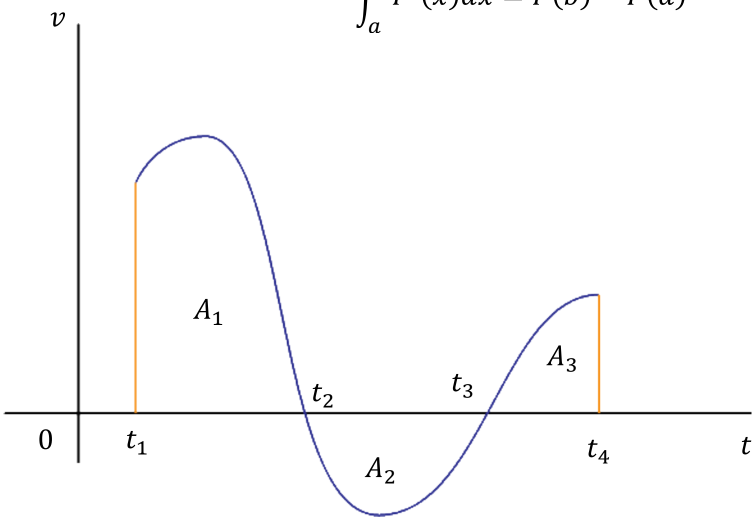
4.19 - Finding Distance Traveled by a Particle Along a Line

The Fundamental Theorem of Calculus can be formulated as:

The Total Change Theorem:

The integral of a rate of change is the total change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$



The figure shows a velocity curve, below is displacement vs. distance interpretation.

Displacement:

$$\int_{t_1}^{t_4} v(t)dt = A_1 - A_2 + A_3$$

Distance:

$$\int_{t_1}^{t_4} |v(t)|dt = \int_{t_1}^{t_2} v(t)dt - \int_{t_2}^{t_3} v(t)dt + \int_{t_3}^{t_4} v(t)dt = A_1 + A_2 + A_3$$

Example:

A particle moves along a curve so that its velocity at time t is $v(t) = t^2 - t - 6$. Find the particles displacement and the total distance travel by it on $[1,4]$.

Displacement

$$\int_1^4 v(t)dt = \int_1^4 (t^2 - t - 6)dt = \frac{t^3}{3} - \frac{t^2}{2} - 6t\bigg]_1^4$$

$$= \left(\frac{4^3}{3} - \frac{4^2}{2} - 24\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right) = -\frac{9}{2}$$

The particle moved 4.5 meters to the left.

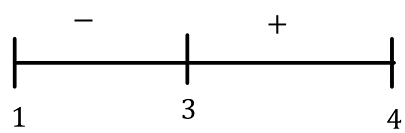
Total distance

$$\int_1^4 |v(t)|dt = \int_1^4 |t^2 - t - 6|dt$$

Now we need to deal with the absolute value

$$t^2 - t - 6 = (t-3)(t+2) = 0$$

$t = 3$ (-2 is outside the interval)



We negate the interval on $[1,3]$:

$$\int_1^4 |t^2 - t - 6|dt = \int_1^3 -(t^2 - t - 6)dt + \int_3^4 (t^2 - t - 6)dt$$

$$= \int_1^3 (-t^2 + t + 6)dt + \int_3^4 (t^2 - t - 6)dt$$

$$= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t\right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_3^4 = \frac{61}{6} \approx 10.71$$

The particle traveled 10.71 meters.

4.20 - Exponential Growth and Decay Model And Solving Differential Equations

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. The **order** of a differential equation is the order of the highest derivative that occurs in the equation. In general the independent variable doesn't have to represent time. For example, when we consider the differential equation

$$y' = xy$$

it is understood that y is a function of x.

A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted into the equation. Thus, f is a solution of the previous equation if

$$f'(x) = xf(x)$$

for all values of f in some interval.

Example:
Show that $y = e^{-2t}$ is a solution to $y'' - 2y' - 8y = 0$.

$$y = e^{-2t}$$

$$y' = -2e^{-2t}$$

$$y'' = 4e^{-2t}$$

$$y'' - 2y' - 8y = 0$$

$$4e^{-2t} - 2(2e^{-2t}) - 8(e^{-2t}) = 0$$

$$4e^{-2t} + 4e^{-2t} - 8e^{-2t} = 0$$

$$e^{-2t}(4 + 4 - 8) = 0$$

$$e^{-2t} \cdot 0 = 0$$

When applying differential equations we are usually not interested in finding a family of solutions (the general solution) as we are in finding a solution that satisfies some additional requirement. In many physical problems we need to find the particular solution that satisfies the a condition of the form $y(t_0) = y_0$. This is called an **initial condition**, and the problem of finding a solution of the differential equation that satisfies the initial condition is called an **initial-value problem**.

Geometrically, when we impose an inital condition, we look at the family of solution curves and pick out the one that passes through the point (t_0, y_0). Physically, this corresponds to measuring the state of a system at time t_0 and using the solution fo the inital-value problem to predict the future behavior of the system.

Example:

Find a solution of the differential equation $y' = \frac{1}{2}(y^2 - 1)$ that satisfies the initial condition $y(0) = 2$. (The general solution is $y = \frac{1+ce^t}{1-ce^t}$).

Substitute the values $t = 0$ and $y = 2$ into the equation:

$$2 = \frac{1 + ce^0}{1 - ce^0} = \frac{1 + c}{1 - c}$$

Then solve for c

$$2 - 2c = 1 + c$$

$$c = \frac{1}{3}$$

So, the solution is

$$y = \frac{1 + \frac{1}{3}e^t}{1 - \frac{1}{3}e^t} = \frac{3 + e^t}{3 - e^t}$$

A **separable equation** is a first-order differential equation in which the expression for $\frac{dy}{dx}$ can be factored as a function of x and a function of y. In other words, it can be written in the form

$$\frac{dy}{dx} = g(x)f(y)$$

The name separable comes from the cast that the expression on the right side can be "separated" into a function of x and a function of y. Evidently, if $f(y) \neq 0$, we could write

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}$$

where $h(y) = \frac{1}{f(y)}$. T solve this equation we rewrite it in the differential form

$$\int h(y)dy = \int g(x)dx$$

In summary, the process for solving a separable differential equation is:

1. Write the equation in the form $\frac{dy}{dx} = \frac{g(x)}{h(y)}$.
2. Rearrange the equation into the form $h(y)dy = g(x)dx$.
3. Integrate both sides
4. Place the constant of integration on the x side(on the right).
5. Solve for y if possible.

Example:

Find the general solution of $y' = \frac{1}{x^2}$.

$$\frac{dy}{dx} = \frac{1}{x^2}$$

$$\int dy = \int \frac{1}{x^2} dx$$

$$y = -\frac{1}{x} + C$$

It can be assumed that a population grows at a rate proportional to the size of the population:

$$\frac{dP}{dt} = kp$$

In general if $y(t)$ is the value of a quantity y at time t and if the rate of change of y with respect to t is proportional to its size $y(t)$ at every t, then $\frac{dy}{dt} = ky$ where k is a constant. This equation is known as the law of natural growth when $k > 0$ and the law of natural decay when $k < 0$; it is also a separable differential equation.

Because it is a separable differential equation, we can solve for y.

$$\frac{dy}{dt} = ky$$

$$\int \frac{dy}{y} = \int k dt$$

$$\ln|y| = kt + C$$

$$|y| = e^{kt+C} = e^C e^{kt}$$

$$y = Ae^{kt}$$

where A ($A = \pm e^C$ or 0) is an arbitrary constant. To see the significance of A, we observe that

$$y(0) = Ae^0 = A$$

Therefore, A is the initial value of the function.

The solution to the initial-value problem

$$\frac{dy}{dt} = ky$$

$$y(0) = y_0$$

$$y(t) = y_0 e^{kt}$$

Example:

A village had a population of 1000 in 1980 and 1200 in 1990. Assuming natural growth, what will the population be in 2010?

$$A = 1000$$

t is years since 1980.

$$y = Ae^{kt}$$

$$y = 1000e^{kt}$$

$$1200 = 1000e^{k(10)}$$

$$y = 1000e^{\left(\frac{\ln(1.2)}{10}\right)(30)}$$

$= 1728$ people

$$\ln(1.2) = 10k$$

$$\frac{\ln(1.2)}{10} = k$$

Other separable differential equations

Radioactive decay

Radioactive substances decay at a rate proportional to the remaining mass. Therefore, the mass decays exponentially:

$$m(t) = m_0 e^{kt}$$

Example:

The half-life of Radium is 1600 years. What is the mass of a 10mg sample of radium after 50 years?

$$m_0 = 10\text{mg}$$

$$t = 50$$

$$m(t) = 10e^{kt}$$

$$5 = 10e^{k(1600)}$$

$$\frac{1}{2} = e^{1600k}$$

$$\ln\left|\frac{1}{2}\right| = 1600k$$

$$\frac{\ln\left(\frac{1}{2}\right)}{1600} = k$$

$$m(50) = 10e^{\left(\frac{\ln\left(\frac{1}{2}\right)}{1600}\right)(50)} \approx 9.788\text{mg}$$

Continuously compounded interest

If an initial amount A_0 is invested at an annual rate of $r\%$ compounded n times a year, the amount after t years is

$$A = A_0\left(1 + \frac{r}{n}\right)^{nt}$$

If the interest in compounded continuously, then we take the limit as $n \to \infty$.

$$\lim_{n\to\infty} A_0\left(1 + \frac{r}{n}\right)^{nt} = A_0 e^{rt}$$

Example:

\$1000 is invested at an interest rate of 4% compounded (a)yearly, (b)monthly, and (c) continuously. After 5 years, how much money is there in the account?

(a) $A = 1000\left(1 + \frac{.04}{1}\right)^{5(1)} = \$1216.65$

(b) $A = 1000\left(1 + \frac{.04}{12}\right)^{5(12)} = \$1221.09$

(c) $A = 1000e^{(.04)(5)} = \$1221.40$

4.21 - Logistic Growth Differential Equation and its Solution

A population often increases exponentially in its early stages but levels off eventually and approaches its carrying capacity because of limited resources. If $P(t)$ is the size of the population at time t, we assume that (when P is small)

$$\frac{dP}{dt} \approx kP$$

This says that the growth rate is initially close to being proportional to size. In other words, the relative growth rate is almost constant when the population is small. But we also want to reflect the fact that the relative growth rate decreases as the population P increases and becomes negative if P ever exceeds its **carrying capacity** M, the maximum population that the environment is capable of sustaining in the long run. The simplest expression for the relative growth rate that incorporates these assumptions is

$$\frac{1}{P}\frac{dP}{dt} = k\left(1 - \frac{P}{M}\right)$$

Multiplying by P we obtain the model for population growth known as the **logistic differential equation**:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

This is a separable differential equation, so we can solve it explicitly. Since

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

we have

$$\int \frac{dP}{P\left(1 - \frac{P}{M}\right)} = \int k dt$$

To evaluate the integral on the right side, we write

$$\frac{1}{P\left(1 - \frac{P}{M}\right)} = \frac{M}{P(M-P)}$$

Using partial fractions, we get

$$\frac{M}{P(M-P)} = \frac{1}{P} + \frac{1}{M-P}$$

We can use this to rewrite the integral

$$\int \frac{1}{P} + \frac{1}{M-P} dP = \int k dt$$

$$\ln|P| - \ln|M-P| = kt + C$$

$$\ln\left|\frac{M-P}{P}\right| = -kt - C$$

$$\left|\frac{M-P}{P}\right| = e^{-kt-C} = e^{-C}e^{-kt}$$

$$\frac{M-P}{P} = Ae^{-kt}$$

where $A = \pm e^{-C}$. Solving this for P we get

$$\frac{M}{P} - 1 = Ae^{-kt} \qquad \Rightarrow \qquad \frac{P}{M} = \frac{1}{1+Ae^{-kt}}$$

$$P = \frac{K}{1+Ae^{-kt}}$$

We find the value of A by putting $t = 0$ into the previous equation ($\frac{M-P}{P} = Ae^{-kt}$). If $t = 0$, then $P = P_0$ (the initial population), so

$$\frac{M-P_0}{P_0} = Ae^0 = A$$

Thus, the solution to the logistic equation is

$$P = \frac{K}{1+Ae^{-kt}} \qquad \text{where } A = \frac{M-P_0}{P_0}$$

Using this expression for $P(t)$, we see that

$$\lim_{t\to\infty} P(t) = M$$

which is to be expected.

Example:

Find the solution to the initial-value problem:

$$\frac{dP}{dt} = .20P\left(1 - \frac{P}{500}\right), \qquad P(0) = 200, \qquad t = \text{years since 1980}$$

What is the population in 1985? When will the population reach 450?

$$A = \frac{500-200}{200} = \frac{3}{2}, \qquad P(t) = \frac{500}{1+\frac{3}{2}e^{-.1t}}$$

$$P = \frac{500}{1+\frac{3}{2}e^{-.1(5)}} = 261.8 \Rightarrow 262$$

$$450 = \frac{500}{1+\frac{3}{2}e^{-.1t}}$$

$$1 + \frac{3}{2}e^{-.1t} = \frac{500}{450}$$

$$\frac{3}{2}e^{-.1t} = \frac{50}{45} - 1$$

$$e^{-.1t} = \frac{1}{9} \cdot \frac{2}{3}$$

$$t = \ln \left|\frac{1}{9}\right| \cdot \ln \left|\frac{2}{3}\right| = 8.909$$

The population is 262 in 1985 and will reach 450 in 1988 (near the end of the year).

4.22 - Arc Length Formulas in Rectangular, Polar, and Parametric Form

Rectangular

The procedure for defining arc length is very similar to the procedure we used for defining area and volume: We divide the curve into a large number of small parts, then find the approximate lengths of the small parts and added them. Finally, we take the limit as $n \to \infty$.

The definition of arc length given by the limit is not very convenient for computational purposes, but we can derive an integral formula for L (arc length) in the case where f has a continuous derivative. (Such an f is called **smooth** because a small change in x produces a small change in $f'(x)$.]

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x), \;\; a \le x \le b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

Parametric

If a curve C is described by the parametric equations $x = f(t), \;\; y = g(t), \;\; \alpha \le x \le \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Polar

To find the length of a polar curve $r = f(\theta), \;\; a \le \theta \le b$, we regard θ as a parameter and write the parametric equations of the curve as

$$x = r\cos\theta = f(\theta)\cos\theta\,, \qquad y = r\sin\theta = f(\theta)\cos\theta$$

Using the product rule and differentiating with respect to θ we obtain

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta\,, \qquad \frac{dy}{d\theta} = \frac{dr}{d\theta\sin\theta} + r\cos\theta$$

so, using $\cos^2\theta + \sin^2\theta = 1$, we have

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 \cos^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta + r^2\sin^2\theta$$

$$+ \left(\frac{dr}{d\theta}\right)^2 \sin^2\theta + 2r\frac{dr}{d\theta}\sin\theta\cos\theta + r^2\cos^2\theta$$

$$= \left(\frac{dr}{d\theta}\right)^2 + r^2$$

Assuming that f' is continuous, we can use the arc length for parametric equations to write the arc length as

$$L = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

Therefore, the length of a curve with polar equation $r = f(\theta),\ \ a \le x \le b$, is

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example:

(a) Find the arc length of $y = x^2$ on $[-1,2]$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$= \int_{-1}^2 \sqrt{1 + (2x)^2} dx$$

$$= 6.126$$

(b) Find the arc length of $y = \ln(\cos x)$ on $\left[0, \frac{\pi}{4}\right]$.

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$= \int_0^{\frac{\pi}{4}} \sqrt{1 + (\tan x)^2} dx$$

$$= .881$$

(c) Find the arc length of the parametric curve $x = t^2,\ \ y = 2t,\ \ 0 \le t \le 2$.

$$\frac{dx}{dt} = 2t, \qquad \frac{dy}{dt} = 2, \qquad [0,2]$$

$$L = \int_0^2 \sqrt{(2t)^2 + (2)^2} dt$$

$$= 5.916$$

(d) Find the arc length of the curve traced out by $r = 2\cos 2\theta$.

$$\frac{dr}{d\theta} = 4\sin 2\theta$$

$$L = \int_0^{2\pi} \sqrt{(2\cos 2\theta)^2 + (4\sin 2\theta)^2} d\theta$$

$$= 19.377$$

5.1 - Geometric Series and Sum of Geometric Series

An important example of an infinite series is the **geometric series**

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} \qquad a \neq 0$$

If $r = 1$, then $s_n = a + a + \cdots + a = na \to \infty$. Since $\lim_{n\to\infty} s_n$ doesn't exist, the geometric series diverges.
If $r \neq 0$, we have

$$s_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

and

$$rs_n = ar + ar^2 + \cdots + ar^{n-1} + ar^n$$

Subtracting these equations, we get

$$s_n - rs_n = a - ar^n$$

$$s_n = \frac{a(1-r^n)}{1-r}$$

If $-1 < r < 1$, we know that $r^n \to 0$ as $n \to \infty$, so

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{a}{1-r} \lim_{n\to\infty} r^n = \frac{a}{1-r}$$

Therefore, the geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1$$

Example:

Prove that the following series is convergent, and find its sum.

$$\sum_{n=0}^{\infty} 5 \cdot \frac{2^n}{3^n} = \sum_{n=0}^{\infty} 5\left(\frac{2}{3}\right)^n$$

$$|r| = \left|\frac{2}{3}\right| < 1$$

therefore, the series is convergent, and its sum is:

$$s_n = \frac{5}{1-\left(\frac{2}{3}\right)} = \frac{5}{\frac{5}{3}} = 3$$

5.2 - P-Series

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $P \leq 1$.

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots$$

$$p = 3 > 1$$

Therefore convergent by p-series test.

Note: The series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent p-series also known as the harmonic series.

5.3 - Alternating Series with Alternating Series Remainder (Error Bound)

An **alternating series** is a series whose terms are alternately positive and negative, usually in the form $a_n = (-1)^{(n-1)} b_n$ or $a_n = (-1)^n b_n$.

If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Note: This only tests for convergence. If the Alternating Series Test fails, use the Test for Divergence.

Alternating Series Estimation Theorem

If $s = \sum(-1)^{n-1} b_n$ is the sum of an alternating series that satisfies

(i) $0 \leq b_{n+1} \leq b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then

$$|R_n| = |s - s_n| \leq b_{n+1}$$

Example:

Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places. (By definition, $0! = 1$.)

(i) $\dfrac{1}{(n+1)!} < \dfrac{1}{n!}$

$n! < (n+1)!$

$n! < n!\,(n+1)$

$1 < n+1$

(ii) $0 < \frac{1}{n!} < \frac{1}{n} \to 0$ so $\frac{1}{n!} \to 0$ as $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{n!} = 0$$

therefore, the series is convergent by alternating series test.

$$\frac{1}{(n+1)!} < .001$$

when $n = 6$,

$$\frac{1}{(6+1)!} = \frac{1}{7!} = \frac{1}{5040} > \frac{1}{5000} = .0002$$

so,

$$s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368 = \text{sum to three decimal places}$$

5.4 - Integral Test

Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words:

(i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.
(ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Example:

Use the integral test to determine convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{n^4}$.

The function $f(x) = \frac{1}{x^4}$ is positive, continuous, and decreasing on $[1, \infty)$, so the integral test can be applied.

$$\int_1^{\infty} \frac{1}{x^4} dx = \lim_{b \to \infty} \int_1^{b} \frac{1}{x^4} dx = \lim_{b \to \infty} \left[\frac{1}{-3(x^3)}\right]_1^b = \lim_{b \to \infty} \left(-\frac{1}{3b^3} + \frac{1}{3}\right) = \frac{1}{3}$$

Because the integral converges, $\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges by the integral test.

5.5 - Ratio Test

(i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L = 1$, then this tests does not determine the convergence of the series $\sum_{n=1}^{\infty} a_n$

Example:

Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

We use the ratio test with $a_n = (-1)^n \frac{n^3}{3^n}$:

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-1)^{n+1}(n+1)^3}{3^{n+1}}}{\frac{(-1)^n n^3}{3^n}}\right| = \frac{(n+1)^3}{3^{n+1}} \cdot \frac{3^n}{n^3} = \frac{1}{3}\left(\frac{n+1}{n}\right)^3 = \frac{1}{3}\left(1+\frac{1}{n}\right)^3 \to \frac{1}{3} < 1$$

Thus, by the ratio test, the given series is absolutely convergent and therefore convergent.

5.6 - Root Test

(i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).

(ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

(iii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = 1$, then this tests does not determine the convergence of the series $\sum_{n=1}^{\infty} a_n$

Example:

Test the convergence of the series $\sum_{n=1}^{\infty} \left(\frac{2n+3}{3n+2}\right)^n$.

$$a_n = \left(\frac{2n+3}{3n+2}\right)^n$$

$$\sqrt[n]{|a_n|} = \frac{2n+3}{3n+2} = \frac{2+\frac{3}{n}}{3+\frac{2}{n}} \to \frac{2}{3} < 1$$

Thus the series converges by the root test.

5.7 - Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Proof

Let m and M be positive numbers such that $m < c < M$. Because $\frac{a_n}{b_n}$ is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N$$

and so

$$mb_n < a < Mb_n \quad \text{when } n > N$$

If $\sum b_n$ converges, so does $\sum Mb_n$. Thus, $\sum a_n$ converges by part (i) of the comparison test. If $\sum b_n$ diverges, so does $\sum mb_n$ and part (ii) of the comparison test shows that $\sum a_n$ diverges.

Example:

Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges of diverges.

We use the limit comparison with $b_n = \sum_{n=1}^{\infty} \frac{1}{2^n}$, a convergent geometric series test ($|r| = \left|\frac{1}{2}\right| < 1$).

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^n}} = 1 > 0$$

Because $\lim_{n\to\infty} \frac{a_n}{b_n}$ exists and $\sum \frac{1}{2^n}$ converges, $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges by the limit comparison test.

5.8 - Direct Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(i) If $\sum b_n$ is convergent and $\sum a_n \leq \sum b_n$ for all n, then $\sum a_n$ is also convergent.
(ii) If $\sum b_n$ is divergent and $\sum a_n \geq \sum b_n$ for all n, then $\sum a_n$ is also divergent.

Proof

(i) Let

$$s_n = \sum_{i=1}^{n} a_i \qquad t_n = \sum_{i=1}^{n} b_i \qquad t = \sum_{n=1}^{\infty} b_n$$

Since both series have positive terms, the sequences $\{s_n\}$ and $\{t_n\}$ are increasing ($s_{n+1} = s_n + a_{n+1} \geq s_n$). Also $t_n \to t$, so $t_n \leq t$ for all n. Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus, $s_n \leq t$ for all n. This means that $\{s_n\}$ is increasing and bounded above and therefore converges by the Monotonic Sequence Theorem. Thus $\sum a_n$ converges.

(ii) If $\sum b_n$ is divergent, then $t_n \to \infty$ (since $\{t_n\}$ is increasing). But $a_i \geq b_i$, so $s_n \geq t_n$. Thus, $s_n \to \infty$. Therefore $\sum a_n$ diverges.

Example:

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ converges or diverges.

Compare it to the series $\sum \frac{5}{2n^2}$ (a multiple of a convergent p-series)

$$\frac{5}{2n^2 + 4n + 3} < \frac{5}{2n^2}$$

$$10n^2 < 10n^2 + 20n + 15$$
$$0 < 20n + 15$$

Therefore $a_n < b_n$, so the series converges by comparison test.

5.9 - Test for Divergence (N-th Term Test)

If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example:

Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2}{5n^2+4} = \lim_{n\to\infty} \frac{1}{5+\frac{4}{n^2}} = \frac{1}{5} \neq 0$$

Therefore the series diverges by n-th term test.

5.10 - Telescoping Series

A **telescoping series** is a series where the terms cancel each other out in pairs. Because of all the cancelations, the sum collapses into just two or three terms. The sum of the series is equal to $\lim_{n\to\infty} s_n$.

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$s_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)}$$

Partial fractal decomposition:

$$\frac{A}{n} + \frac{B}{n+1} = \frac{1}{n(n+1)}$$

n	$A(n+1)+Bn$	1
-1	$0+-B$	$B=-1$
0	$A+0$	$A=1$

$$= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1} = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{n}+\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

So

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} 1 - \frac{1}{n+1} = 1 - 0 = 1$$

Therefore, the series is a telescoping series and converges by the telescoiping series test. The sum of the series is 1.

5.11 - Taylor Polynomials

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \qquad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Substituting this formula for c_n into the series, we see that if f has a power series expansion at a, then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

This series is called the **Taylor series of the function f at a** (or **about a** or **centered on a**). For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

This case arises often enough that it is given the special name **Maclaurin series**.

The **nth-degree Taylor polynomial of f at a**, T_n is equal to

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note: The Taylor polynomial is a finite set of terms, while the Taylor series is an infinite series.

Example:

Find the Maclaurin series for e^x and the Taylor polynomial at 0 (or Maclaurin polynomial) with $n = 3$.

$$f(x) = e^x$$

$$f^{(n)}(x) = e^x$$

$$f^{(n)}(0) = e^0 = 1$$

Therefore, the Taylors series of f at 0 is

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

and the 3rd degree Taylor polynomial is

$$e^x \approx T_3(x) = \sum_{i=0}^{3} \frac{f^{(i)}(x)}{i!} (x-a)^i = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

5.12 - Power Series for Elementary Functions

Maclaurin Series for e^x:

$$f(0) = e^x = e^0 = 1$$

$$f'(0) = e^x = e^0 = 1$$

$$f''(0) = e^x = e^0 = 1$$

$$f'''(0) = e^x = e^0 = 1$$

$$f^{(4)}(0) = e^x = e^0 = 1$$

$$f(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad (-\infty, \infty)$$

Maclaurin Series for $\sin x$:

$$f(0) = \sin 0 = 0$$

$$f'(0) = \cos 0 = 1$$

$$f''(0) = -\sin 0 = 0$$

$$f'''(0) = -\cos 0 = -1$$

$$f^{(4)}(0) = \sin 0 = 0$$

$$f(x) = \frac{0x^0}{0!} + \frac{1x^1}{1!} + \frac{0x^2}{2!} - \frac{x^3}{3!} + \frac{0x^4}{4!} + \cdots$$
$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad (-\infty, \infty)$$

Maclaurin Series for $\cos x$:

$$f(0) = \cos 0 = 1$$

$$f'(0) = -\sin 0 = 0$$

$$f''(0) = -\cos 0 = -1$$

$$f'''(0) = \sin 0 = 0$$

$$f^{(4)}(0) = \cos 0 = 1$$

$$f(x) = \frac{x^0}{0!} + \frac{0x^1}{1!} - \frac{x^2}{2!} + \frac{0x^3}{3!} + \frac{x^4}{4!} + \cdots$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad (-\infty, \infty)$$

Maclaurin Series for $\frac{1}{1-x}$:

$$f(x) = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad (-1,1)$$

This is the sum formula for a geometric series ($s = \frac{a}{1-r}$).

Maclaurin Series for $\tan^{-1} x$:

$$f(0) = \arctan x = 0$$

$$f'(0) = \frac{1}{1+x^2} = 1$$

$$f''(0) = -2x(1+x^2)^{-2} = 0$$

$$f(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7}$$

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad [-1,1]$$

Maclaurin Series for $\ln(1+x)$:

$$f(0) = \ln(1+0) = 0$$

$$f'(0) = \frac{1}{1+0} = 1$$

$$f''(0) = -1(1+0)^{-2} = -1$$

$$f'''(0) = 2(1+0)^{-3} = 2$$

$$f^{(4)}(0) = -3!\,(1+0)^{-4} = -3!$$

$$f(x) = \frac{0x^0}{0!} + \frac{1x^1}{1!} - \frac{1x^2}{2!} + \frac{2x^3}{3!} - \frac{3!\,x^4}{4!}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}, \qquad (-1,1]$$

5.13 - How to Find Radius and Interval of Convergence

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

The number R in case (iii) is called the **radius of convergence** of the power series. By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii). The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges. In case (i) the interval consists of just a single point a. In case (ii) the interval is $(-\infty, \infty)$. In case (iii) note that the inequality $|x-a| < R$ can be rewritten as $a - R < x < a + R$. When x is an endpoint of the interval, that is $x = \pm R$, anything can happen - the series might converge at one or both endpoints or it might diverge at both endpoints. Thus, in case (iii) there are four possibilities for the interfal of convergence:

$$(a-R, a+R), \qquad (a-R, a+R], \qquad [a-r, a+R), \qquad [a-R, a+R]$$

To find the radius and integral of convergence, the ratio test must be applied to the power series and then the endpoints tested for convergence [if case (iii)].

Example:

Find the radius and integral of convergence for:

$$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}$$

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)(x+2)^{n+1}}{3^{n+2}}}{\frac{n(x+2)^n}{3^{n+1}}} \right| = \lim_{n\to\infty} \left[\frac{(n+1)(x+2)^{n+1}}{3^n \cdot 3^2} \cdot \frac{3^n \cdot 3}{n(x+2)^n} \right]$$

$$= \lim_{n\to\infty} \frac{n+1}{3n} \cdot |x+2|$$

$$= \lim_{n\to\infty} \frac{\frac{n}{n} + \frac{1}{n}}{\frac{3n}{n}} |x+2| = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \left|\frac{x+2}{3}\right| = \left|\frac{x+2}{3}\right| < 1$$

$$|x+2| < 3$$

Radius of convergence: 3

Integral of convergence:

$$-3 < x + 2 < 3$$

$$-5 < x < 1$$

$$= (-5, 1)$$

when $x = -5$:

$$\sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n n$$

which diverges by the n-th term test.

when $x = 1$:

$$\sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3}\sum_{n=0}^{\infty} n$$

which diverges by the n-th term test.

Thus, the series only converges when $-5 < x < 1$, so the interval of convergence is $(-5,1)$.

5.14 - Lagrange Error Bound for Taylor Polynomials

In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x)$$

If we let

$$R_n(x) = f(x) - T_n(x) \qquad \text{so that} \qquad f(x) = T_n(x) + R_n(x)$$

then $R_n(x)$ is called the **remainder** of the Taylor series. If we can somehow show that $\lim_{n\to\infty} R_n(x) = 0$, then it follows that

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x)$$

We have therefore proved the following.

If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a and

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a| < R$, then f is equal to the sum of its Taylor series on the interval $|x-a| < R$.

In trying to show that $\lim_{n\to\infty} R_n(x) = 0$ for a specific function f, we usually use the following fact (Taylor's Inequality).

If $\left|f^{(n+1)}(x)\right| \le M$ for $|x-a| \le d$, then the remainder $\mathrm{R_n}(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for } |x-a| \le d$$

Example:

(a) Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

(b) How accurate is this approximation when $7 \le x \le 9$?

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \qquad f(8) = 2$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \qquad f'(8) = \frac{1}{12}$$

$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} \qquad f''(8) = -\frac{1}{144}$$

$$f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}$$

Thus, the second degree Taylor polynomial is

$$T_2(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-2)^2$$

$$= 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 2)^2$$

The desired approximation is

$$\sqrt[3]{x} \approx T_2(x) = 2 + \frac{1}{12}(x - 8) - \frac{1}{288}(x - 2)^2$$

The Taylor series is not alternating when $x < 8$, so we can't use the Alternating Series Estimation Theorem in this example. But we can use Taylor's Inequality with $n = 2$ and $a = 8$:

$$|R_2(x)| \leq \frac{M}{3!}|x - 2|^3$$

where $|f'''(x)| \leq M$. Because $x \geq 7$, we have $x^{8/3} \geq 7^{8/3}$ and so

$$f'''(x) = \frac{10}{27} \cdot \frac{1}{x^{\frac{8}{3}}} \leq \frac{10}{27} \cdot \frac{1}{7^{\frac{8}{3}}} < 0.0021$$

Therefore, we can take $M = 0.0021$. Also $7 \leq x \leq 9$, so $-1 \leq x - 8 \leq 1$ and $|x - 8| < 1$. Then Taylors Inequality gives

$$|R_2(x)| \leq \frac{0.0021}{3!} \cdot 1^3 = \frac{0.0021}{6} = 0.0004$$

Thus, if $7 \leq x \leq 9$, the approximation is accurate to within 0.0004.